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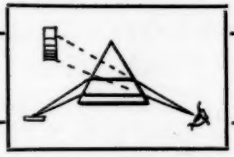

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COMPARISON OF WRIGHT'S DATA ON EQUIVALENT COLOR STIMULI WITH THE O. S. A. DATA*

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ABSTRACT

Wright's results constitute the only published data on equivalent color stimuli that have arisen from the use of modern apparatus. Previous data (Abney, König) refer to apparatus such that only approximate results are to be expected; these results are embodied in the O. S. A. "excitation" curves which are widely used in America as the standard basis for colorimetric computation. A comparison is presented of Wright's experimental results with the O. S. A. data. It is shown that a considerable portion of the differences between Wright's experimental results and the O. S. A. "excitation" curves are such as would be caused by an increase in macular pigmentation; that is, it would seem that Wright's results refer to a somewhat heavier macular pigmentation. The advantages of supplanting the O. S. A. "excitation" curves with a new set of distribution curves embodying Wright's experimental results, and also the advantages of retaining the O. S. A. "excitation" curves as standard for colorimetric computation are discussed. It is concluded that these procedures are about equally likely to be advantageous.

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I. INTRODUCTION

Since 1922, colorimetric computation in America has been carried out mostly by means of the so-called O. S. A. "excitation" curves.[1] These curves embody in a very simple form a weighted mean of observations on equivalent color stimuli taken by König and Dieterici earlier than 1890[2] and by Abney[3] during the next two decades. Even though the number of observes (three) is too limited to evaluate reliably an average, normal, visual mechanism, and even though the apparatus used was such that only approximately correct results could be anticipated, the derivation and publication of the O. S. A. "excitation" curves have been of great value and utility. This utility is attested by the fact that they have been adopted by international agreement for the standardization of colored signal glasses.[4] As the standard representation of the properties of the average, normal, visual mechanism they have, however, three outstanding defects: (1) the curves are unspecified for wave-length regions for which a specification of zero is obviously in error, (2) the curves exhibit irregularities which it would seem could be smoothed without significantly affecting their meaning,[5] and (3) there are no weights

[1] L. T. Troland, Report of Committee on Colorimetry for 1920–21, J.O.S.A. & R.S.I., *6*, 547–553; 1922.

[2] A. König and C. Dieterici, *Die Grundempfindungen und ihre Intensitätsvertheilung im Spectrum*, Sitz. Akad. Wiss., Berlin, 805–829; 1886; or see A. König, *Gesammelte Abhandlungen zur Physiologischen Optik*, Leipzig, Barth, 60–87; 1903. *Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsvertheilung im Spectrum*, ZS. f. Psych. u. Physiol. der Sinnesorgane, *4*, 241–347; 1892; or see A. König, Ges. Abh., 214–321.

[3] W. de W. Abney, *Researches in Colour Vision and the Trichromatic Theory*, London, Longmans-Green, 246; 1913.

[4] Proc. International Commission on Illumination, 7th meeting, Saranac Inn, N. Y., 14; September, 1928.

[5] As, for example, was done by Guild (*The Geometrical Solution of Colour Mixture Problems*, Trans. Opt. Soc., *26*, 163–165; 1924–25) preparatory to the adoption of a similar reduction of partly identical data.

which, when attached to the curves, will make them sum exactly to the standard visibility function. It is true that the first defect has been largely remedied by an extrapolation by Priest and Gibson[6] and that the second defect is serious only with respect to the annoyance and loss of time occasioned by attempts to make values obtained by interpolation with one interpolation graph agree with values similarly obtained with another. Furthermore, it has been shown[7] that the discrepancies resulting from the third defect are negligible in comparison to individual differences in visibility; hence we may characterize these discrepancies as annoying and time-wasting rather than really serious.

It might well be supposed that the three defects mentioned are entirely subsidiary to the question of whether the mean of three questionable sets of data on equivalent color stimuli referring to but three observers is a sufficiently reliable characterization of average, normal vision to merit adoption as standard for colorimetric computation. In particular, it might appear unwise to attach much importance to results by only three observers in view of the well-known wide variations in vision among supposedly normal observers; but, as a matter of fact, for the purpose of colorimetric computation it is not necessary to use the characteristics of a statistically determined, average observer. The very fact which serves to raise this question robs it of importance; that is, the wide variation among normal observers renders it largely a matter of indifference whether we have in hand the statistical mean or merely curves of the same general character. It would be nearly as futile to hope to find an average, normal observer as it is to hope to find a single observer who embodies the composite characteristics of König, Dieterici and Abney as evaluated by the O. S. A. "excitation" curves. The normality of the mechanism characterized by a standard set of distribution curves for use in colorimetric computation is really, then, only a matter of convenience; and lack of normality causes in practice considerably less inconvenience and annoyance than any of the defects listed above.

In support of this conclusion it may be remarked that the O. S. A. "excitation" curves have served, since 1922, many useful purposes during which time scepticism concerning their accuracy has been quite general. These purposes include the solution of such problems in equiva-

[6] Spectrophotometry; Report of O.S.A. Progress Committee for 1922–3, J.O.S.A. & R.S.I., *10*, 230; 1925. In the present paper these extrapolated values are used.

[7] D. B. Judd, *Reduction of Data on Mixture of Color Stimuli*, B. S. Jour. Research, *4*, 524–527; 1930. Cf. also D. B. Judd, *Chromatic Visibility Coefficients by the Method of Least Squares*, J.O.S.A. & R.S.I., *10*, 635–651; 1925.

lence of color stimuli as are relatively unaffected by the usual variations in color sense, for example, the comparison of two different beams of radiant energy whose spectral distributions are fairly similar. By computing on the basis of the O. S. A. "excitation" curves the trilinear coordinates corresponding to the colors evoked by the two beams it is often possible to reduce the difference between the two stimuli to parameters (such as dominant wave length, colorimetric purity, or color temperature) in terms of which normal sensibility is known, and hence to estimate the degree of difference. Of course, cases arise in which failure to represent the vision of a given observer becomes serious,—notably cases of color match between two stimuli of widely differing spectral distribution of energy. But it is impossible to find two observers agreeing on such matches except in isolated cases and the discrepancy is usually much larger than the uncertainty of determination by either of the observers. Hence, it is plainly not to be expected that a single set of curves for use in routine colorimetric computation will ever apply strictly to more than one observer. The failure, then, of the O. S. A. "excitation" curves to give a faithful representation of average, normal vision introduces only such inconveniences as would be shared as well by any other single set of curves.

It would appear, therefore, that the O. S. A. "excitation" curves have been valuable, not because they give a faithful description of the average, normal observer, but because they give a description of a fairly representative observer to which, by general agreement, all stimuli are referred for the sake of uniformity. That the observer described be fairly normal is essential; whether it be a statistical norm is of considerable but secondary importance; this importance arises largely because of a perfectly natural inclination to avoid a set of curves unless that set be the best that can be compiled. But it is not necessary that the representation be within certain limits in order that the curves be adopted; the representation need only be the best obtainable, since we must, perforce, adopt some set of curves.

In 1922, the O. S. A. "excitation" curves were at least very nearly the best set obtainable. The recent determinations of Wright,[8] however, have marked so definite a step forward that the O. S. A. "excitation" curves do not now constitute the most faithful description obtainable

[8] W. D. Wright, *A Re-Determination of the Trichromatic Coefficients of the Spectral Colours*, Trans. Opt. Soc., *30*, 141–164; 1928–29; *A Re-Determination of the Trichromatic Mixture Data*, Medical Research Council, Reports of the Committee upon the Physiology of Vision, VII, Special Report Series No. 139, London, 1929.

of the average, normal eye. Wright has determined with modern apparatus[9] for ten observers the relative proportions of lights of wave lengths, 460, 530, and 650 mμ respectively, required to color match the other portions of the spectrum. He found marked individual differences in these proportions when the results were expressed (as is usual) in such units that equal amounts of the primaries yield a match for "white" light (the National Physical Laboratory standard white, a close chromaticity match for the Planckian radiator at 4,800°K).[10] The major amounts of these individual differences were regular, however, and could be eliminated by a change of units for each observer. With this new choice of units, of course, the N. P. L. standard white could not be matched by equal proportions of the primaries, but by some other proportions; hence, the "white" point of such a colorimetric system would not fall at the center of the Maxwell triangle, but at some other point. Wright has very plausibly ascribed the major amounts of the individual differences to difference in macular pigmentation;[11] the distribution of "white" points on the Maxwell triangle corroborated this view. The smaller part of the individual differences which could not be eliminated in this way were taken as indications either of variations in the actual receptor sensitivities from one observer to another or of experimental error. In addition to the complete data throughout the spectrum for the 10 observers, the macular pigmentation of 36 observers was investigated by an abbreviation of this procedure. The final results give the mean for the ten observers in such terms that the average color of the standard white as seen by 36 observers is represented at the center of the Maxwell triangle. In this way, Wright has accomplished the separation of the individual differences introduced by the assumed variations in pigmentation of the eye media from the individual differences due to all other causes. The significance of the average results of the ten observers is considerably easier to comprehend than would be the case if the separation had not been accomplished; in particular it is apparent that

[9] W. D. Wright, *A Trichromatic Colorimeter with Spectral Primaries*, Trans. Opt. Soc., *29*, 225–242; 1927–28.

[10] To be designated hereafter as the "N. P. L. standard white."

[11] J. C. Maxwell, *On the Theory of Compound Colours and the Relations of the Colours of the Spectrum*, Phil. Trans., 57; 1860; also Maxwell, Scientific Papers, *1*, p. 434. Such an interpretation is also in agreement with a more recent conclusion by Sinden (*Studies based on Spectral Complementaries*, J.O.S.A. and R.S.I., *7*, 1138–1151; 1923) that the most important source of discrepancy in vision among normal observers is difference in macular pigmentation. Sinden, to eliminate this source of discrepancy, suggested the use of a different yellow filter for each observer.

Wright's mean experimental results are much more likely to represent average, normal vision than are the O. S. A. "excitation" curves.

As already pointed out, the mere availability of a more faithful representation of average, normal vision is little practical justification for supplanting the O. S. A. data; however, considerable annoyance and extra work in the application of the curves in colorimetric computation could be avoided if the defects (2) and (3) already mentioned were remedied. Hence, if the O. S. A. "excitation" curves are to be supplanted anyway by a set of curves which do not possess defects (2) and (3), it would seem natural to consider carefully a set of curves embodying Wright's experimental results.

It is the purpose of the present paper: (1) to derive a set of distribution curves exhibiting Wright's data in a form more directly comparable to the present standard set known as the O. S. A. "excitation" curves, and (2) to present the arguments for and against the adoption of a new set of distribution curves to supplant the present standard set.

II. DEFINITIONS AND STANDARDS

1. GENERAL

In discussing the derivation of distribution curves comparable to the O. S. A. "excitation" curves from Wright's experimental results, it is convenient to adopt terms different from those used by Wright, and, indeed, a somewhat more elaborate system is required than has heretofore seemed necessary. In general, we follow the symbology introduced by Priest several years ago[12] and used by the present author in a previous paper[13] which employs the symbols, ρ, γ, β, to represent (as functions of wave length) the ordinates of the three distribution curves. A set of distribution curves is not adequately designated, however, unless four factors are identified: (a) the data on equivalent color stimuli which are represented, (b) the primary color processes whose distributions throughout the spectrum are given, (c) the distribution of energy in the spectrum to which the curves refer, and (d) the relative scale numerals of the three curves.

To identify the data on equivalent color stimuli represented by a given set of distribution curves it is merely necessary to state the source

[12] I. G. Priest, *The Computation of Colorimetric Purity*, J.O.S.A. & R.S.I., *9*, 503–520; 1924.

[13] D. B. Judd, *Reduction of Data on Mixture of Color Stimuli*, B. S. Jour. Research, *4*, 515–548; 1930.

of the data; that is, one would identify the observer or group of observers represented by the curves.

The primary color processes may be identified by giving the stimuli (real or imaginary) which serve to evoke them. Such stimuli may be specified by spectral distributions of energy; the real stimuli by distributions involving only ordinates greater than or equal to zero; the imaginary stimuli, by distributions involving some ordinates less than zero. Another method of dealing with imaginary primaries may also be used; it is discussed later. (See section III, step 4.)

The relative scale numerals of the three curves are conveniently specified by giving the distribution of energy of a stimulus whose color is arbitrarily represented at the center of the Maxwell triangle. Such a stimulus has been named the "basic stimulus" by Priest.[14]

It is proposed to identify each set of distribution curves herein treated by identifying each of these four factors with a subscript; that is, ρ, γ, β will each be written with four subscripts, the first subscript serving to identify the first factor; the second, the second, etc. For example, the O. S. A. "excitation" curves referred to in the introduction[15] would be designated ρ_{oooo}, γ_{oooo}, β_{oooo}, where the first subscript, o, indicates that the mixture data are those adopted by the O. S. A., the second subscript, o, indicates that the primary color processes are those adopted by the O. S. A., the third subscript, e, indicates that the curves refer to an equal energy spectrum, and the fourth subscript, o, indicates that the basic stimulus is O. S. A. mean noon sun. For another example, the set of distribution curves against which Wright's data are compared in the present paper are indicated by ρ_{ooaa}, γ_{ooaa}, β_{ooaa}, the third subscript, a, meaning that the curves refer to the spectrum of Abbot-Priest sunlight[16] and the fourth subscript, a, meaning that the scale numerals of the three distribution curves have been so adjusted that the color of Abbot-Priest sunlight is represented at the center of the Maxwell triangle. For a third example, the curves embodying Wright's data on equivalent color stimuli in such a way as to be conveniently comparable to ρ_{ooaa}, γ_{ooaa}, β_{ooaa} would be designated as ρ_{woaa}, γ_{woaa}, β_{woaa}. The first subscript, w, means that the data represented are those of Wright. The

[14] See Priest's discussion of Guild's paper, *On the Fixed Points of a Colorimetric System*, Optical Society (London) Meeting of February 12, 1931.

[15] See footnotes 1 and 6, pp. 700 and 701.

[16] D. B. Judd, *Reduction of Data on Mixture of Color Stimuli*, B. S. Jour. Research, *4*, 524–525; 1930; see particularly footnote 37, and column 1 of Table 1.

identity of the last three subscripts in the two cases indicates the comparability of the two sets of curves.

The methods of transforming sets of distribution curves from one choice to another among the last three factors are well known and are not given here in the general case. The actual equations of transformation used in the special cases treated are given in the next section.

Throughout the paper it is convenient to make use of some definite discriminatory (or "sensation") increment in describing chromaticity differences. The one chosen may be called the "least perceptible difference under usual observing conditions," where "usual observing conditions" may be taken as observation by one eye of a field subtending 2 or 3° with a dark surrounding field. The size of this chromaticity difference corresponds closely to the size of the Jones "hue step."[17] It is a rather liberal "least perceptible difference," however, because chromaticity differences of one-fifth or one-tenth this size may be readily detected by a single scrutiny under particularly favorable observing conditions.

2. THE ADOPTED STANDARDS OF VISIBILITY

In the comparisons to be presented, two standards of visibility are adopted, V_o and V_s. The former is the visibility embodied in the O. S. A. "excitation" curves for given luminosity coefficients; it is defined as:

$$V_0 \equiv (0.45\rho_{\omega aa} + 0.54\gamma_{\omega aa} + 0.01\beta_{\omega aa})/E_a$$

where E_a is the spectral distribution of energy for Abbot-Priest sunlight.[18] The latter, V_s, is the standard visibility function.[19] By combining the data on equivalent color stimuli obtained by Wright with V_o we assume temporarily that the visibility function of the O. S. A. "excitation" curves adequately represents Wright's observers. The interest in making this assumption is that it enables us to discover how much the O. S. A. "excitation" curves would have to be changed in order to agree with Wright's data on equivalent color stimuli; that is, the comparison

[17] L. A. Jones, *The Fundamental Scale of Pure Hue and Retinal Sensibility to Hue Differences*, J.O.S.A., *1*, 63–77: 1917. The "Jones hue scale" also appears in the Report of the Colorimetry Committee of the O. S. A., 1920–21 (J.O.S.A. and R.S.I., *6*, 545; 1922) under the name "spectral chroma scale," and in a later publication by Guild (*A Critical Survey of Modern Developments in the Theory and Technique of Colorimetry and Allied Sciences*, Proc. Optical Convention, Part I, 1926) as his Fig. 8 (p. 47) and his Table V (p. 48).

[18] See footnote 16, p. 705.

[19] K. S. Gibson and E. P. T. Tyndall, *Visibility of Radiant Energy*, B. S. Sci. Paper No. 475, 174; 1923. The visibility function taken is that recommended by these authors. It was subsequently adopted as standard by the International Commission on Illumination, 6th meeting, Geneva, July 1924; Proceedings, pp. 67 and 232.

between the O. S. A. "excitation" curves and Wright's mixture data is made as direct as possible.

By combining the data on equivalent color stimuli obtained by Wright with V_s, we assume that the standard visibility function adequately represents Wright's observers. The interest in making this assumption is that it enables us to derive a set of distribution curves which probably represents more closely the colorimetric characteristics of the average, normal eye than do the O. S. A. "excitation" curves.

When the first assumption is being considered, the first subscript to the symbols representing the distribution curves will be written, w_o. This will indicate that Wright's mixture data have been combined with the visibility of the O. S. A. "excitation" curves. When the second assumption is being considered, the first subscript will be written, w_s. This will indicate that Wright's mixture data have been combined with the standard visibility function. Whenever the first subscript is written, w, without any secondary subscript, either of these assumptions is referred to, in which case the visibility will also be designated, V, without subscript.

3. THE SPECTRAL DISTRIBUTION OF ENERGY FOR WRIGHT'S BASIC STIMULUS

The basic stimulus used by Wright is the N. P. L. standard white specified by the chemical composition of two liquid filters and the color temperature of a source. It is said to be a close chromaticity match for a Planckian radiator at 4,800°K. From the chemical composition of the filters and from the manner of its establishment the spectral distribution of energy of the N. P. L. standard white has been deduced by Davis and Gibson.[20] This energy distribution departs rather considerably from that for a Planckian radiator at 4,800°K, but this departure, of course, is of no consequence for the use of the N. P. L. standard white as basic stimulus. However, a comparison of the trilinear coordinates of the color corresponding to this evaluation of the N. P. L. standard white with those of the Planckian locus on the Maxwell triangle indicates that the N. P. L. standard white is too greenish to match any Planckian color exactly. It could scarcely be argued that this difference is serious, since it is estimated to be about one-fifth of a least perceptible difference under ordinary conditions of observation.[21] However, in order to check the dependence of ρ_{woaa}, γ_{woaa}, β_{woaa} on this slight uncertainty in specifi-

[20] R. Davis and K. S. Gibson, *Spectral Energy Distribution and Correlated Color Temperature of the N.P.L. Standard White Light*, B. S. Jour. Research, 7, 791–797; 1931.

[21] See footnote 17, p. 706.

cation of basic stimulus we assume first the spectral distribution of energy, E_n', deduced by Gibson and Davis for the basic stimulus, and second we assume the distribution of energy, E_n'', for a Planckian radiator at 4,800°K for $C_2 = 14{,}350$ micron degrees as the basic stimulus. When the first subscript to the symbols representing the distribution curves is written, w', the first assumption will be meant; for the second assumption we write, w''. Whenever the first subscript is written, w, without any primes, either of these assumptions is referred to. Whenever the subscript, n, appears in third or fourth place, the N. P. L. standard white is referred to, the choice between these two assumptions being designated by the first subscript as just described.

III. Derivation, from Wright's Data, of Distribution Curves Conveniently Comparable with the O. S. A. "Excitation Curves"

The desired curves are designated: ρ_{woaa}, γ_{woaa}, β_{woaa}. They are to be compared to: ρ_{ooaa}, γ_{ooaa}, β_{ooaa}.[22] They are found by four steps of computation, each step resulting in distribution curves designated as follows:

Step 1, ρ_{wwnn}, γ_{wwnn}, β_{wwnn};
Step 2, ρ_{wwan}, γ_{wwan}, β_{wwan};
Step 3, ρ_{wwaa}, γ_{wwaa}, β_{wwaa};
Step 4, ρ_{woaa}, γ_{woaa}, β_{woaa}.

Step 1. Wright's average experimental results are given as the "trichromatic coefficients," R, G, B, referred to the N. P. L. standard white as basic stimulus. In our notation R, G, B would be defined:

$$\left.\begin{aligned} R &\equiv \rho_{wwnn}/(\rho_{wwnn} + \gamma_{wwnn} + \beta_{wwnn}) \\ G &\equiv \gamma_{wwnn}/(\rho_{wwnn} + \gamma_{wwnn} + \beta_{wwnn}) \\ B &\equiv \beta_{wwnn}/(\rho_{wwnn} + \gamma_{wwnn} + \beta_{wwnn}) \end{aligned}\right\} \qquad (1)$$

and would be called the trilinear coordinates of the spectrum according to this particular set of distribution curves. Wright also gives approximate values of the luminosity coefficients of his primary color processes, $(L_r)_{wwnn}$, $(L_g)_{wwnn}$, $(L_b)_{wwnn}$, which, for the purpose of the present comparison, are taken such that:

$$\rho_{wwnn}(L_r)_{wwnn} + \gamma_{wwnn}(L_g)_{wwnn} + \beta_{wwnn}(L_b)_{wwnn} \equiv CE_nV \qquad (2)$$

[22] This transformation of the O. S. A. "excitation" curves has already been published; see footnote 13, p. 704.

where C is an arbitrary constant.

From (1) and (2) it follows that:

$$\left.\begin{aligned} \rho_{wwnn} &\equiv \frac{CRE_nV}{R(L_r)_{wwnn} + G(L_g)_{wwnn} + B(L_b)_{wwnn}} \\ \gamma_{wwnn} &\equiv \frac{CGE_nV}{R(L_r)_{wwnn} + G(L_g)_{wwnn} + B(L_b)_{wwnn}} \\ \beta_{wwnn} &\equiv \frac{CBE_nV}{R(L_r)_{wwnn} + G(L_g)_{wwnn} + B(L_b)_{wwnn}} \end{aligned}\right\} \quad (3)$$

and by substituting as we choose either E_n' or E_n'' for E_n, and either V_o or V_s for V, and Wright's approximate values for the luminosity coefficients, $(L_r)_{wwnn}$, $(L_g)_{wwnn}$, $(L_b)_{wwnn}$, we may obtain approximately the curves of step 1 for one of the cases of interest. This set of curves is next checked against the condition that:

$$\int_0^\infty \rho_{wwnn}d\lambda = \int_0^\infty \gamma_{wwnn}d\lambda = \int_0^\infty \beta_{wwnn}d\lambda \quad (4)$$

to discover whether the N. P. L. standard white is the basic stimulus for ρ_{wwnn}, γ_{wwnn}, β_{wwnn} in conformity with the specification of R, G, B by Wright. In none of the three cases is condition (4) found to be exactly satisfied. This is interpreted to mean that the approximate experimental values of the luminosity coefficients found by Wright are not exactly congruent with the visibility function assumed. We then alter the values of the luminosity coefficients in such a way as to satisfy more nearly condition (4). By trial and error in this way the true values of the luminosity coefficients are found[23] in which case relation (3) gives the curves of step 1.

Step 2. Refer the curves of step 1 to the spectrum of Abbot-Priest sunlight, E_a, instead of to the spectrum of the N. P. L. standard white, E_n, thus:

$$\left.\begin{aligned} \rho_{wwan} &\equiv (E_a/E_n)\rho_{wwnn} \\ \gamma_{wwan} &\equiv (E_a/E_n)\gamma_{wwnn} \\ \beta_{wwan} &\equiv (E_a/E_n)\beta_{wwnn} \end{aligned}\right\}. \quad (5)$$

Step 3. The areas beneath the curves defined by (5) are not equal as is the case for the curves defined by (3) since the chromaticity of Abbot-Priest sunlight is different from that of the N. P. L. standard white. Curves having equal areas are therefore computed:

[23] See appendix I. The trial and error method was continued in each of the three cases computed until condition (4) was satisfied to within one part in a thousand.

$$\left.\begin{aligned}
\rho_{wwaa} &\equiv \left(1{,}000{,}000 \Big/ \int_0^\infty \rho_{wwan} d\lambda\right)\rho_{wwan} \\
\gamma_{wwaa} &\equiv \left(1{,}000{,}000 \Big/ \int_0^\infty \gamma_{wwan} d\lambda\right)\gamma_{wwan} \\
\beta_{wwaa} &\equiv \left(1{,}000{,}000 \Big/ \int_0^\infty \beta_{wwan} d\lambda\right)\beta_{wwan}
\end{aligned}\right\}. \qquad (6)$$

As indicated by the fourth subscript, the basic stimulus for these curves is Abbot-Priest sunlight.

The integrals of relation (6) are evaluated with sufficient approximation by computing: $\sum_0^\infty \rho_{wwan}\Delta\lambda$, etc., for $\Delta\lambda = 10$ mμ. It is to be noted that the curves computed by relation (6) not only have equal areas but all total to the definite sum, 1,000,000. This number is chosen because the curves, ρ_{eeaa}, γ_{eeaa}, β_{eeaa}, with which comparison is to be made, all total to it.

Step 4. The primary color processes whose distributions are given by the curves found in step 3 are those corresponding to the spectrum stimuli of wave length, 650, 530, and 460 mμ, respectively. As a final step in making Wright's data comparable to the O. S. A. data, it is desired to transform these curves so that they refer to the O. S. A. primaries which are extreme spectrum red, extreme spectrum violet, and a green more saturated than spectrum green. The green primary process corresponds to an imaginary stimulus; that is, the stimulus can not be specified by a spectral distribution of energy unless negative values of energy at some wave lengths be admitted. Instead of dealing with energy distributions of this sort, we carry out as has been done before[24] a transformation merely ensuring that the trilinear coordinates of three suitably chosen spectrum colors be equal to those for the same respective wave lengths according to ρ_{eeaa}, γ_{eeaa}, β_{eeaa}. Define:

$$\left.\begin{aligned}
\rho_{weaa} &\equiv K_1\rho_{wwaa} + K_2\gamma_{wwaa} + K_3\beta_{wwaa} \\
\gamma_{weaa} &\equiv K_4\rho_{wwaa} + K_5\gamma_{wwaa} + K_6\beta_{wwaa} \\
\beta_{weaa} &\equiv K_7\rho_{wwaa} + K_8\gamma_{wwaa} + K_9\beta_{wwaa}
\end{aligned}\right\} \qquad (7)$$

the constants, K_1 to K_9, being chosen so as to yield equality in trilinear coordinates at 410, 530, and 670 mμ and also so as to retain Abbot-

[24] J. Guild, *The Transformation of Trichromatic Mixture Data: Algebraic Method*, Trans. Opt. Soc., *26*, 95–108; 1924–25. D. B. Judd, *The Mixture Data Embodied in the Tentative Curves of Hecht's Theory of Vision*, J.O.S.A., *20*, 647–660; 1930.

Priest sunlight as the basic stimulus.[25] These are the desired curves; and these four steps of computation were carried out for three separate cases resulting in the following three sets of distribution curves:

$$\rho_{w_0''oaa},\ \gamma_{w_0''oaa},\ \beta_{w_0''oaa};$$

$$\rho_{w_0'oaa},\ \gamma_{w_0'oaa},\ \beta_{w_0'oaa};$$

$$\rho_{w_s'oaa},\ \gamma_{w_s'oaa},\ \beta_{w_s'oaa}.$$

By combining relations (3), (5), (6), and (7) and writing S_{wnn} as an abbreviation for $R(L_r)_{wnn}+G(L_g)_{wnn}+B(L_b)_{wnn}$, it is found that:

$$\left.\begin{aligned}
\rho_{woaa} &= \frac{1{,}000{,}000\,CE_aV}{S_{wnn}}\left[\frac{K_1R}{\int_0^\infty \rho_{wwan}d\lambda}+\frac{K_2G}{\int_0^\infty \gamma_{wwan}d\lambda}+\frac{K_3B}{\int_0^\infty \beta_{wwan}d\lambda}\right]\\
\gamma_{woaa} &= \frac{1{,}000{,}000\,CE_aV}{S_{wnn}}\left[\frac{K_4R}{\int_0^\infty \rho_{wwan}d\lambda}+\frac{K_5G}{\int_0^\infty \gamma_{wwan}d\lambda}+\frac{K_6B}{\int_0^\infty \beta_{wwan}d\lambda}\right]\\
\beta_{woaa} &= \frac{1{,}000{,}000\,CE_aV}{S_{wnn}}\left[\frac{K_7R}{\int_0^\infty \rho_{wwan}d\lambda}+\frac{K_8G}{\int_0^\infty \gamma_{wwan}d\lambda}+\frac{K_9B}{\int_0^\infty \beta_{wwan}d\lambda}\right]
\end{aligned}\right\}. \quad (8)$$

These relations (8) may be used to check the accuracy of the computation of ρ_{woaa}, γ_{woaa}, β_{woaa} which were actually found in each case by way of the three intermediate sets of curves. These relations also serve as a condensed indication of the connection of ρ_{woaa}, γ_{woaa}, β_{woaa} to Wright's experimental results, R, G, B, and the visibility function, V. We can not actually dispense with the three intermediate sets of curves, however; the first set, ρ_{wnn}, γ_{wnn}, β_{wnn}, must be computed in order to test by condition (4) whether values for $(L_r)_{wnn}$, $(L_g)_{wnn}$, $(L_b)_{wnn}$ have been correctly chosen;[26] the second set must be computed in order to evaluate the integrals which appear in the denominators of the expressions of relation (8); and the third set, ρ_{waa}, γ_{waa}, β_{waa}, must be at least partially evaluated in order to obtain the constants, K_1 to K_9, of relation (7). Similarly the spectral distribution of energy, E_n, of the N. P. L.

[25] The general methods of choosing these constants for various purposes have long been known. Furthermore, a detailed account of a very similar problem has already been given (D. B. Judd, *The Mixture Data Embodied in the Tentative Curves of Hecht's Theory of Vision*, J.O.S.A., *20*, 647–660; 1930) and the arguments and limitations given according to which this procedure is equivalent to the expression of one set of data in terms of the primaries of another.

[26] See appendix II.

standard white cancels out and does not appear in relation (8); but it is necessary to know E_n, otherwise $(L_r)_{wwnn}$, $(L_g)_{wwnn}$, $(L_b)_{wwnn}$ could not be evaluated (relations (3) and (4)).

IV. Discussion of Results

Table 1 gives, for the three cases computed, the constants of relation (8) by means of which ρ_{woaa}, γ_{woaa}, β_{woaa}[27] become known functions of the mean experimental values, R, G, B, reported by Wright. In addition

TABLE 1. *Constants used in computation and checking of* ρ_{woaa}, γ_{woaa}, β_{woaa}.

Designation of constant	E_n taken as for Plackian radiator at 4,800°K	E_n taken as deduced by Davis and Gibson from the characteristics of the filters and light source by means of which the N.P.L. standard white is specified	
	V taken as V_o, the visibility for the O. S. A. "excitation" curves		V taken as V_s, the standard visibility
	w_o''	w_o'	w_s'
C	1.0116	1.0832	101.75
$(L_r)_{wwnn}$	0.2719	0.26958	0.26926
$(L_g)_{wwnn}$	.6983	.70074	.70468
$(L_b)_{wwnn}$	.0298	.02968	.02606
$\int_0^\infty \rho_{wwaa} d\lambda/10^6$	.96431	1.03713	1.03637
$\int_0^\infty \gamma_{wwaa} d\lambda/10^6$	1.02839	1.09905	1.09904
$\int_0^\infty \beta_{wwaa} d\lambda/10^6$	1.04017	1.12762	1.10964
K_1	.50228	.50239	.50177
K_2	.50157	.50116	.50187
K_3	−.00388	−.00355	−.00364
K_4	.00708	.00703	.00649
K_5	.94813	.94734	.94869
K_6	.04479	.04563	.04482
K_7	.00052	.00052	.00048
K_8	.07003	.06997	.07007
K_9	.92945	.92951	.92945
$(L_r)_{woaa}$	.50948	.50702	.50750
$(L_g)_{woaa}$	.47844	.48145	.48517
$(L_b)_{woaa}$	.01208	.01154	.00734

[27] The three cases are indicated in Table 1 by labels at the heads of the columns and also by giving there the first of the corresponding four subscripts, thus: w_o'', w_o', w_s'.

to the constants of relation (8) we add in Table 1 the values of $(L_r)_{woaa}$, $(L_g)_{woaa}$, $(L_b)_{woaa}$,[28] which are useful in a closing check to determine whether the visibility, V, has really been incorporated in ρ_{woaa}, γ_{woaa}, β_{woaa} according to the relation:

$$\rho_{woaa}(L_r)_{woaa} + \gamma_{woaa}(L_g)_{woaa} + \beta_{woaa}(L_b)_{woaa} = E_aV,$$

where E_aV is, of course, expressed in such scale numerals as make $\int_0^\infty E_aV d\lambda = 1{,}000{,}000$, since, as is customary, the luminosity coefficients have been adjusted so as to sum to unity. This gives E_aV_o directly as inserted in relation (8), but gives E_aV_s of relation (8) times a constant (93.985). The closing check afforded by these luminosity coefficients has been carried out for the three cases computed and it was found that the discrepancies so disclosed are ascribable to errors of rejection in the computation.

Table 2 gives ρ_{ooaa}, γ_{ooaa}, β_{ooaa} together with the corresponding trilinear coordinates of the spectrum, R_{ooaa}, G_{ooaa}, B_{ooaa}.[29] Tables 3, 4, and 5 as indicated in the headings give ρ_{woaa}, γ_{woaa}, β_{woaa} for each of the three cases computed[30] together with the corresponding trilinear coordinates of the spectrum.

Fig. 1 compares ρ_{ooaa}, γ_{ooaa}, β_{ooaa}, given in Table 2, with $\rho_{w_o'oaa}$, $\gamma_{w_o'oaa}$, $\beta_{w_o'oaa}$, given in Table 4; that is, Fig. 1 compares the O. S. A. "excitation" curves for Abbot-Priest sunlight as basic stimulus with a set of distribution curves throughout the same spectrum embodying Wright's experimental results for the same primaries, for the same basic stimulus, and for the same visibility function. The O. S. A. "excitation" curves are apparently not much different from $\rho_{w_o'oaa}$, $\gamma_{w_o'oaa}$, $\beta_{w_o'oaa}$; and, indeed, as reference to Tables 3, 4, and 5 shows, the red and green curves of the other two cases which have been computed ($\rho_{w_o''oaa}$, $\gamma_{w_o''oaa}$, and $\rho_{w_s'oaa}$, $\gamma_{w_s'oaa}$, see Tables 3 and 5) so nearly resemble the case illustrated here that the differences would scarcely appear on the scale we have chosen. It may be noted, however, that a large part of the small difference between the ρ and γ curves compared in Fig. 1 could be taken up by a shift of about 2 mμ in the wave-length scale. The β curves could be brought into closer agreement by an equal shift

[28] Found from the other constants of Table 1 by methods previously described. See footnote 13, p. 704.

[29] The definition of these trilinear coordinates of the spectrum as well as those given in Tables 3, 4, and 5 is analogous to definition (1).

[30] The totals of the columns headed, ρ, γ, β, in Tables 2 to 5 are also given. These totals differ from 100,000 (cf. step 3 of preceding section) because of rejection error in computation.

TABLE 2. *The O. S. A. "excitation" curves (extrapolated) for the spectrum of Abbot-Priest sunlight and for Abbot-Priest sunlight as basic stimulus; also the corresponding trilinear coordinates of the spectrum.*

λ in mμ	ρ_{0000}	γ_{0000}	β_{0000}	R_{0000}	G_{0000}	B_{0000}
380			442			
390			1,274			
400			2,717			
410			5,045	0.000	0.000	1.000
420		13	7,679	.000	.002	.998
430		41	12,186	.000	.003	.997
440		100	14,351	.000	.007	.993
450		240	14,027	.000	.017	.983
460	15	595	12,958	.001	.044	.955
470	64	1,313	11,098	.005	.105	.890
480	231	2,040	7,767	.023	.203	.774
490	693	2,897	3,703	.095	.397	.508
500	1,430	4,546	2,112	.177	.562	.261
510	2,644	6,944	1,517	.238	.625	.137
520	4,125	9,160	1,076	.287	.638	.075
530	5,479	10,357	765	.330	.624	.046
540	6,681	10,958	517	.368	.604	.028
550	7,604	11,134	322	.399	.584	.017
560	8,325	10,475	196	.438	.551	.011
570	8,989	9,336	124	.487	.506	.007
580	9,210	7,456	71	.550	.446	.004
590	9,388	5,269	35	.639	.359	.002
600	8,841	3,447	17	.719	.280	.001
610	7,900	1,960		.801	.199	.000
620	6,291	1,004		.862	.138	.000
630	4,690	484		.906	.094	.000
640	3,161	164		.951	.049	.000
650	1,871	48		.975	.025	.000
660	1,057	16		.985	.015	.000
670	606			1.000	.000	.000
680	326					
690	187					
700	105					
710	54					
720	33					
Totals	100,000	99,997	99,999			

TABLE 3. *Distribution curves throughout the spectrum of Abbot-Priest sunlight embodying: (1) Wright's experimental data assuming the N. P. L. standard white to match a Planckian radiator at 4,800°K, (2) the visibility function of the O. S. A. "excitation" curves, (3) the primary color processes of the O. S. A. "excitation" curves, and (4) Abbot-Priest sunlight as basic stimulus. The corresponding trilinear coordinates of the spectrum are also given; see relation(1).*

λ in mμ	ρ_{w_0}''oaa	γ_{w_0}''oaa	β_{w_0}''oaa	R_{w_0}''oaa	G_{w_0}''oaa	B_{w_0}''oaa
380			305			
390			1,097			
400			2,255			
410			4,120	0.000	0.000	1.000
420	− 6	7	6,908	−.001	.001	1.000
430	−15	44	10,840	−.001	.004	.997
440	−43	145	12,593	−.003	.011	.992
450	−65	309	12,999	−.005	.023	.982
460	−58	669	13,882	−.004	.046	.958
470	113	1,339	12,418	.008	.096	.895
480	367	2,080	8,340	.034	.193	.773
490	815	3,003	5,015	.092	.340	.568
500	1,633	4,705	2,993	.175	.504	.321
510	2,932	7,182	1,939	.243	.596	.161
520	4,437	9,479	1,241	.293	.625	.082
530	5,696	10,767	795	.330	.624	.046
540	6,870	11,330	546	.366	.604	.029
550	7,863	11,341	438	.400	.577	.022
560	8,620	10,466	325	.444	.539	.016
570	9,300	9,084	302	.498	.486	.016
580	9,432	7,032	226	.565	.421	.014
590	9,282	4,888	201	.646	.340	.014
600	8,571	3,083	125	.728	.262	.011
610	7,436	1,726	65	.806	.187	.007
620	5,826	848	23	.870	.127	.003
630	4,338	341	4	.926	.073	.001
640	2,866	111		.963	.037	.000
650	1,684	24		.986	.014	.000
660	944	6.8		.993	.007	.000
670	534	0.0		1.000	.000	.000
680	289	−1.6		1.006	−.006	.000
690	166	−1.6		1.010	−.010	.000
700	94	−1.0		1.011	−.011	.000
710	48	−0.4				
720	30	−0.2				
Totals	99,999	100,005	99,995			

TABLE 4. *Distribution curves throughout the spectrum of Abbot-Priest sunlight embodying: (1) Wright's experimental data assuming the N. P. L. standard white to match the color corresponding to the spectral distribution of energy deduced by Davis and Gibson, (2) the visibility function of the O. S. A. "excitation" curves, (3) the primary color processes of the O. S. A. "excitation" curves, and (4) Abbot-Priest sunlight as basic stimulus. The corresponding trilinear coordinates of the spectrum are also given; see relation (1).*

λ in mμ	$\rho_{w_0'oaa}$	$\gamma_{w_0'oaa}$	$\beta_{w_0'oaa}$	$R_{w_0'oaa}$	$G_{w_0'oaa}$	$B_{w_0'oaa}$
380			284			
390			1,149			
400			2,325			
410			4,270	0.000	0.000	1.000
420	− 3	9	6,985	.000	.001	.999
430	−14	47	11,033	−.001	.004	.997
440	−41	150	12,708	−.003	.012	.991
450	−61	313	12,906	−.005	.024	.981
460	−53	675	13,749	−.004	.047	.957
470	120	1,345	12,284	.009	.098	.893
480	372	2,076	8,201	.035	.195	.770
490	817	2,993	4,933	.093	.342	.564
500	1,634	4,689	2,947	.176	.506	.318
510	2,930	7,161	1,914	.244	.597	.159
520	4,430	9,457	1,230	.293	.626	.081
530	5,685	10,747	794	.330	.624	.046
540	6,856	11,315	549	.366	.604	.029
550	7,847	11,333	445	.400	.577	.023
560	8,604	10,469	332	.443	.540	.017
570	9,284	9,095	307	.497	.487	.016
580	9,419	7,049	232	.564	.422	.014
590	9,280	4,909	205	.645	.341	.014
600	8,578	3,098	127	.727	.262	.011
610	7,447	1,736	67	.805	.188	.007
620	5,840	854	25	.869	.127	.004
630	4,350	344	3	.926	.073	.001
640	2,876	111		.963	.037	.000
650	1,690	24		.986	.014	.000
660	948	6.7		.993	.007	.000
670	537	0.0		1.000	.000	.000
680	291	−1.6		1.006	−.006	.000
690	166	−1.9		1.012	−.012	.000
700	94	−1.5		1.016	−.016	.000
710	46	−0.7				
720	29	−0.2				
Totals	**99,998**	**100,000**	**100,004**			

TABLE 5. *Distribution curves throughout the spectrum of Abbot-Priest sunlight embodying: (1) Wright's experimental data assuming that the N. P. L. standard white matches the color corresponding to the spectral distribution of energy deduced by Davis and Gibson, (2) the Gibson-Tyndall recommended visibility function adopted as the international standard, (3) the primary color processes of the O. S. A. "excitation" curves, and (4) Abbot-Priest sunlight as basic stimulus; the corresponding trilinear coordinates of the spectrum are also given, see relation (1).*

λ in mμ	$\rho_{w_s\text{'oaa}}$	$\gamma_{w_s\text{'oaa}}$	$\beta_{w_s\text{'oaa}}$	$R_{w_s\text{'oaa}}$	$G_{w_s\text{'oaa}}$	$B_{w_s\text{'oaa}}$
380			25.7			
390			86.7			
400			316.5			
410			1,032	0.000	0.000	1.000
420	− 2.3	3.9	3,593	−.001	.001	1.000
430	−13	40	9,570	−.001	.004	.997
440	−52	177	15,552	−.003	.011	.992
450	−89	439	18,414	−.005	.023	.981
460	−68	833	17,278	−.004	.046	.958
470	115	1,301	12,078	.009	.096	.895
480	362	2,025	8,126	.034	.193	.773
490	808	2,960	4,950	.093	.340	.568
500	1,558	4,470	2,847	.176	.504	.321
510	2,778	6,790	1,836	.244	.595	.161
520	4,357	9,301	1,218	.293	.625	.082
530	5,780	10,926	807	.330	.624	.046
540	6,993	11,542	555	.366	.605	.029
550	7,908	11,423	443	.400	.578	.022
560	8,668	10,551	327	.443	.540	.017
570	9,232	9,049	299	.497	.487	.016
580	9,480	7,095	230	.564	.422	.014
590	9,308	4,922	202	.645	.341	.014
600	8,581	3,096	125	.727	.262	.011
610	7,425	1,727	65	.806	.187	.007
620	5,921	861	23	.870	.127	.003
630	4,283	335	4	.927	.072	.001
640	2,892	109		.964	.036	.000
650	1,768.2	22.9		.987	.013	.000
660	995.5	6.0		.994	.006	.000
670	512.3	−0.4		1.001	−.001	.000
680	267.7	−1.7		1.006	−.006	.000
690	125.8	−1.6		1.013	−.013	.000
700	61.3	−0.7		1.012	−.012	.000
710	30.5	−0.4				
720	14.65	−0.20				
Totals	100,001	100,000	100,003			

in the opposite direction. It is possible to account for some of this difference by the assumption of a different degree of macular pigmentation between the two sets of distribution curves; this possibility is discussed presently. The other outstanding differences have to do with the β curves; the irregularity near the maximum of $\beta_{w_o'oaa}$ is to be traced to an irregularity in the visibility function of ρ_{ooaa}, γ_{ooaa}, β_{ooaa}, that is, in V_o. This irregularity has been imposed on $\rho_{w_o'oaa}$, $\gamma_{w_o'oaa}$, $\beta_{w_o'oaa}$ as the secondary subscript, o, indicates; and, of course, it appears by far the more prominently in $\beta_{w_o'oaa}$ because of the relatively greater magnitude of the curve within this wave-length region (430 to 460 mμ).

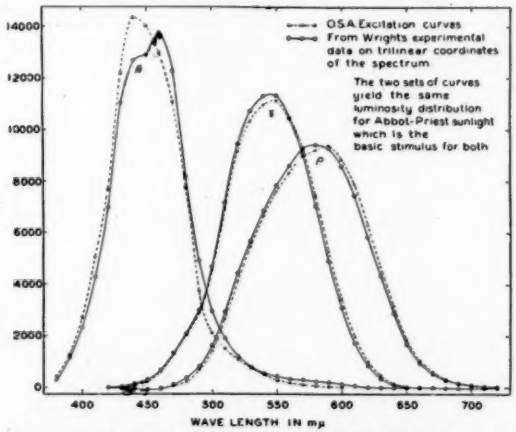

FIG. 1. *Comparison of the O. S. A. "excitation" curves (Table 2) with a set of distribution curves (Table 4) combining Wright's data on equivalent color stimuli with the visibility data of the O. S. A. "excitation" curves. Wright's data are taken as if the N. P. L. standard white were a chromaticity match for the spectral distribution of energy deduced for it by Davis and Gibson.*

Fig. 2 shows the same two β curves as Fig. 1, β_{ooaa}, $\beta_{w_o'oaa}$, and, in addition, shows $\beta_{w_s'oaa}$ for comparison. It may be seen that the irregularity near the maximum of $\beta_{w_o'oaa}$ is not duplicated in $\beta_{w_s'oaa}$ which embodies the standard visibility function, V_s; this serves to corroborate the above conclusion that the irregularity near the maximum of $\beta_{w_o'oaa}$ in Fig. 1 is due to irregularity of the visibility function embodied in it. Note, however, that the excess of $\beta_{w_o'oaa}$ over β_{ooaa} for wave length greater than 540 mμ can not be ascribed to irregularity in visibility function, since it also appears for $\beta_{w_s'oaa}$.

Since the two sets of curves compared in Fig. 1 both yield the same visibility function, the differences between them can not be said to

refer to differences in visibility; these differences must, therefore, indicate differences in the proportions of primary stimuli required to match the various spectrum colors. From Fig. 1 we can not, however, estimate

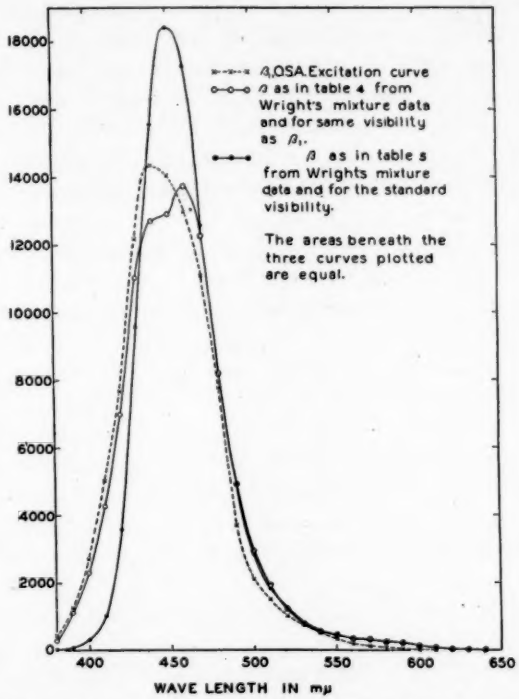

FIG. 2. *Comparison of three "blue" distribution curves. Note that the curve referring to standard visibility is much lower between 380 and 420 mμ than the curve referring to the visibility of the O. S. A. "excitation" curves. Note also that the irregularity near the maximum of the curve which results from combining Wright's data with the visibility of the O. S. A. "excitation" curves does not appear in the curve referring to the standard visibility.*

whether the differences are important or negligible. It has been empirically found, on the other hand, for these primary color processes and for this basic stimulus that a difference in r-coordinate or a difference in g-coordinate of 0.010 corresponds to one least perceptible difference under usual observing conditions[31] to within a factor of 2.[32] By a

[31] See footnote 17, p. 706.

[32] D. B. Judd, *Saturation of Colors Determined from the Visual Response Functions*, J.O.S.A. & R.S.I., *16*, 115; 1928. This reference cites an abstract; the complete paper is in preparation.

comparison of R_{ooaa} and G_{ooaa} in Table 2 with R_{woaa} and G_{woaa}, respectively (Table 3, 4, or 5), we can determine the approximate importance of the discrepancies. It is seen that the largest discrepancies occur in the

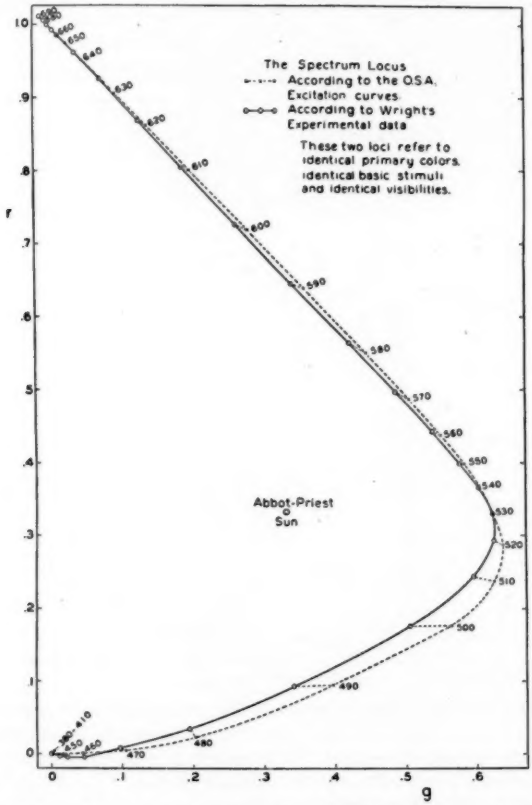

FIG. 3. *Comparison of the spectrum locus according to Wright's data on equivalent color stimuli (Table 4) with the spectrum locus according to the O. S. A. "excitation" curves (Table 2). The greatest discrepancies between the two loci occur at 490 and at 500 mu where it is estimated that they amount to about six least perceptible differences under ordinary conditions of observation. These discrepancies are not large compared to individual differences among normal observers.*

g-coordinate for 490 and 500 mμ; these discrepancies in trilinear coordinates of about 0.060 correspond to approximately 6 least perceptible differences under usual observing conditions. These discrepancies are shown graphically in Fig. 3 in which the spectrum locus (dashed

line) according to ρ_{ooaa}, γ_{ooaa}, β_{ooaa} is compared to the spectrum locus (solid line) according to $\rho_{w_0'oaa}$, $\gamma_{w_0'oaa}$, $\beta_{w_0'oaa}$ by plotting R against G. The points representing the colors corresponding to the same wave lengths are connected by straight dotted lines which serve to display at a glance the nature and magnitude of the discrepancies between the two loci. It will be noted that the two loci coincide at 410, 530, and 670 mμ; this coincidence is imposed by the choice of constants, K_1 to K_9 (see Table 1 and relations (7) and (8)), as one of the conditions required in order to render the two sets of curves comparable. The major part of the discrepancies between 490 and 530 mμ are such as would be expected between two observers of differing macular pigmentation; that is, the agreement between the two sets of curves for these wave lengths would be considerably improved if the O. S. A. "excitation" curves were changed so as to refer to observation through a yellow filter similar to the average human macula according to data by Sachs.[33] Agreement for other wave lengths would be relatively little affected though, in general, made somewhat poorer. Since the variation found by Wright among his 36 observers which was ascribed to difference in macular pigmentation is about 4 times this magnitude, it would seem not unreasonable to suppose that the mean of Wright's 36 observers corresponds to a macular pigmentation heavier by about the amount measured by Sachs. Obviously the mean of 36 observers is a more faithful indication of average macular pigmentation than the O. S. A. "excitation" curves which are a mean referring to three observers so weighted as to make interpretation difficult. The remaining differences may be roughly described as an increased curvature in spectrum locus from 510 to 530 mμ. These differences must be ascribed to individual difference in receptor sensitivity and to experimental error, the results by Wright having by far the greater claim to reliability in both these respects as well as in respect to individual difference in macular pigment. Since, however, none of the discrepancies are large compared to individual differences, it may be said that Wright's results in this sense check the O. S. A. "excitation" curves, and give them a degree of importance and reliability which they have hitherto not possessed.

By intercomparing the r-coordinates and g-coordinates of the spectrum, R_{woaa}, G_{woaa}, given in Tables 3, 4, and 5, it may be discovered that the spectrum loci by all three sets of distribution curves calculated

[33] M. Sachs, Ueber die specifische Lichtabsorption des gelben Fleckes der Netzhaut, Pflüger's Arch. f. d. ges. Physiol., *50*, 574–586; 1891.

so as to embody Wright's experimental results are nearly identical; that is, the differences are of the order of 0.001 which corresponds to about 1/10 of a least perceptible difference for ordinary observing conditions. In a few cases the differences are 0.002, such as, for example, at 490 and 500 mμ. By this comparison between Tables 3 and 4 ($R_{w_9''oaa}$ with $R_{w_9'oaa}$, $G_{w_9''oaa}$ with $G_{w_9'oaa}$) is indicated that the uncertainty in specification of the N. P. L. standard white is not large enough to cause concern in its use as basic stimulus by Wright; the uncertainty would seem likely to be in every case smaller than one-fifth of a least perceptible difference. Indeed, slightly larger differences are introduced by a change in visibility function (compare $R_{w_9'oaa}$ with $R_{w_s'oaa}$, and $G_{w_9'oaa}$ with $G_{w_s'oaa}$, Tables 4 and 5) by an amount entirely negligible in comparison to individual differences in visibility.[34]

V. THE QUESTION OF ADOPTING A NEW SET OF STANDARD DATA

It is evident that the set of distribution curves, $\rho_{w_s'oaa}$, $\gamma_{w_s'oaa}$, $\beta_{w_s'oaa}$, (Table 5) possesses properties which would seem to make it a worthy candidate for adoption as the standard in colorimetric computation to supplant the O. S. A. "excitation" curves. These properties may be summarized as follows:

1. They represent the degree of macular pigmentation corresponding to an average of 36 observers.
2. They represent the mixture data corresponding to an average of 10 observers, each of which yields nearly identical results when the individual differences due to macular pigmentation have been eliminated.
3. They represent exactly the standard visibility function.
4. They are sufficiently regular that interpolation of good reproducibility is possible.

The first two properties listed constitute superiorities over the O. S. A. "excitation" curves which, however, are probably insufficient to justify the adoption of the new set.[35] The argument in favor of supplanting the O. S. A. "excitation" curves is that those curves sufficiently fail to possess properties (3) and (4) that some revision would be expedient anyhow; we may, then, with advantage also incorporate properties (1) and (2) in the revision.

It is possible, of course, to discover arguments opposing the supplant-

[34] The two visibility functions in question have already been graphically compared; see footnote 13, p. 704.

[35] Cf. preceding section and the introduction.

ing of the O. S. A. "excitation" curves by the set of distribution curves specified in Table 5. Some of these arguments are:

1. The irregularity of the O. S. A. "excitation" curves which makes reproducible interpolation difficult ought to be met either by small changes in the curves, themselves, which would smooth out the irregularities[36] or by the adoption of interpolated values at close intervals along the wave-length scale, say at every millimicron.[37] In either of these cases, while perfect continuity would not exist between old and new computed values, the discrepancies would not be important because they would be of the order of the present uncertainty in interpolation. On the other hand, the adoption of a set of curves embodying Wright's results would bring a discontinuity large enough to prevent significant intercomparison of old and new computations.

2. The only remaining argument for supplanting the O. S. A. "excitation" curves is that they do not embody the standard visibility function, but the discrepancies using suitable luminosity coefficients[38] are so small that no one claims they are serious; the claim is rather that the small discrepancies cause annoyance and inconvenience. But it will be annoying and inconvenient to change to a new set of curves for colorimetric computation; and it is a question whether the gain in convenience which might ultimately be available is worth all the trouble which is certain to attend the change. It would seem wise to wait until the arguments for revision of the O. S. A. "excitation" curves are more urgent.

3. The new set of distribution curves embodying Wright's results on equivalent color stimuli yield a spectrum locus which is concave outward. See Fig. 3, 560 to 630 mμ on the solid locus.[39] Such a concavity would be interpreted as indicating that a mixture of two spectrum stimuli could produce a stimulus that would appear to be purer than the spectrum stimulus of the same dominant wave length; that is, certain spectral distributions of energy exist which correspond to stimuli of colorimetric purity greater than unity according to Wright's experimental results. Since this phenomenon has never been experimentally demonstrated it has been customary to regard as in error any set of dis-

[36] See footnote 5, p. 700.

[37] Such interpolated values are now available: D. B. Judd, Interpolation of the O. S. A. "Excitation" Data by the Fifth-Difference Osculatory Formula, J.O.S.A., *21*, 531–540; 1931.

[38] See footnote 7, p. 701.

[39] This concavity of spectrum locus should be evident from any linear, homogeneous transformation of Wright's experimental results. For example, it may be seen in Wright's own graphs of the spectrum locus; see footnote 8, p. 702.

tribution curves which implies the existence of this phenomenon.[40] It would seem wise, therefore, to postpone the revision or supplantation of the O. S. A. "excitation" curves by this set of distribution curves until this question is resolved.

4. The new set of distribution curves embodying Wright's results have a few specified values less than zero in spite of the fact that they agree in trilinear coordinates at 410, 530, and 670 mμ with the O. S. A. values which are all positive or zero. These negative values constitute an annoying complication in the use of the data for colorimetric computation.

The last argument (4) is pertinent but of little weight because the introduction of the negative values is not necessary. In accord with this argument it would be better to choose for routine computation a slightly different coordinate system in order to eliminate these negative values. In fact, for this purpose it might be advantageous as previously suggested[41] to adopt the coordinate system which results from taking the luminosity function as one of the distribution curves.

Argument (3) may also be regarded lightly. The amount of concavity of the spectrum locus is quite slight (about one-fifth of a least perceptible difference under ordinary conditions of observation as may be seen from Fig. 3) and is readily ascribable to experimental error. Although it might have been somewhat preferable to impose the condition that the spectrum locus must be nowhere concave outward in taking the mean of the results by the ten observers, the amount of concavity Wright found by taking the mean without restrictions is small enough to be safely neglected; indeed it is of the same order of magnitude as the uncertainty in specification of the N. P. L. standard white which has previously been shown to be negligible in this connection. Hence, the concavity of the spectrum locus according to Wright's data is not a defect at all serious.

The first argument (1), however, defending the O. S. A. data from rejection on the ground of uncertainties of interpolation is probably sound.

The real contention for adopting the new data would seem to lie in argument (2) having to do with failure of the O. S. A. "excitation" curves to embody exactly the standard visibility function. Whether the

[40] E. Schrödinger, Über das Verhältnis der Vierfarben- zur Dreifarbentheorie, Sitz. Akad. Wiss., Wien, IIa, *134*, 479; 1925. See also footnote 25, p. 711.

[41] See footnote 13, p. 704.

gain in convenience due to the correction of this defect is worth the trouble of making it, is hard to estimate; plainly the judgment depends on how long Wright's results may be expected to dominate available mixture data as they do now; that is, if sufficient refinement in experimental methods comes into use within 5 or 10 years to raise anew the question of standard data it might be better to wait during that interval and make a single revision instead of two; but if Wright's apparatus and technic are not going to be materially improved upon within, say, the next 100 years, then the sooner the O. S. A. "excitation" curves are supplanted, the better. These two procedures seem to be about equally promising.

VI. APPENDIX

1. APPENDIX I

An aid to the computation of luminosity coefficients and an indirect check on the normal visibility function

An empiric relation has been found which facilitated what might otherwise have taken many days of trail computation,—the evaluation of $(L_r)_{wwnn}$, $(L_g)_{wwnn}$, $(L_b)_{wwnn}$ by trail and error according to relations (2), (3), and (4). Let L_r', L_g', L_b' be the best available approximate values of the luminosity coefficients; compute by relation (3) three functions of the wave length, ρ', γ', β', which will satisfy condition (4) well or ill depending upon whether the approximations to the luminosity coefficients are good or bad; then the following relations give closely the true luminosity coefficients:

$$K(L_r)_{wwnn} = L_r'\left[1 + 2.00\left(\frac{\sum\rho'\Delta\lambda}{\sum\gamma'\Delta\lambda} - 1\right)\right]\left[1 - 0.10\left(\frac{\sum\beta'\Delta\lambda}{\sum\gamma'\Delta\lambda} - 1\right)\right]$$

$$K(L_b)_{wwnn} = L_b'\left[1 + 0.62\left(\frac{\sum\beta'\Delta\lambda}{\sum\gamma'\Delta\lambda} - 1\right)\right]\left[1 - 0.10\left(\frac{\sum\rho'\Delta\lambda}{\sum\gamma'\Delta\lambda} - 1\right)\right]$$

$$K(L_g)_{wwnn} = L_g'$$

where K is a constant inserted for the purpose of making the luminosity coefficients sum to unity as is usual. K may be found as the sum of the right members of these three relations. The indicated sums are approximate evaluations of integrals similar to those of condition (4); see also section III, step 3. The constants of these relations as well as the relations, themselves, were found empirically; so it will be recognized that the luminosity coefficients found by them will not satisfy condition (4)

exactly. Two successive applications of these relations, however, usually proved to be sufficient to yield values of the luminosity coefficients such that condition (4) was satisfied within one part in a thousand. Naturally these relations refer only to the three primaries used by Wright; other choices of primaries would necessitate different constants and, perhaps, a different form of relation.

It is worth noting that $(L_r)_{w_o''wnn}$, $(L_g)_{w_o''wnn}$, $(L_b)_{w_o''wnn}$ were found by this method promptly in spite of the fact that, due to an oversight, the trial values, L_r', L_g', L_b', originally chosen differed from the true values rather widely. L_r', L_g', L_b' were originally chosen as 0.3877, 0.5822, 0.0302 which seemed to be in conformity with the average of Wright's seven observers.[42] However, it was discovered later by reference to the more complete account[43] that these values refer to distribution curves of such relative magnitudes that the N. P. L. standard white is represented at r-coordinate = 0.243, g-coordinate = 0.410 rather than at the center of the Maxwell triangle (r-coordinate = g-coordinate = $\frac{1}{3}$) as condition (4) necessitates. Wright's experimental values for the luminosity coefficients expressed so as to refer to the N. P. L. standard white as basic stimulus should have been taken; these are found to be: 0.2744, 0.6951, 0.0305, which agree much better with the values finally found by the method of trial and error: 0.2719, 0.6983, 0.0298, see Table 1. This agreement, in fact, is well within the individual variation found by Wright; it indicates that the visibility of the O. S. A. "excitation" curves for the luminosity coefficients chosen (see definition of V_o, section II, part 2) is consistent with the visual characteristics of Wright's observers.

It is of interest to note also that the luminosity coefficients, $(L_r)_{w_o'wnn}$, $(L_g)_{w_o'wnn}$, $(L_b)_{w_o'wnn}$, referring to the standard visibility function (see Table 1) do not agree so well with the experimentally determined values just given, particularly $(L_b)_{w_o'wnn}$ which is found to be 0.02606. This value is lower than the value found by six of Wright's seven observers. This would indicate that in so far as V_o is higher for short wave length than the standard visibility function, V_s, Wright's observers agree with it better than with the standard visibility function. This evidence that the standard visibility function is too low for short wave lengths is rather slight compared to that on which the standard visibility is based,

[42] Trans. Opt. Soc., *30*, 156; 1928–29.

[43] Medical Research Council, Reports of the Committee upon the Physiology of Vision, VII, Special Report Series No. 139, London, 31; 1929.

but it serves to corroborate in this respect the experimental results of Gibson and Tyndall.[44]

2. APPENDIX II

Alternative methods of obtaining the luminosity coefficients

It has been suggested by Guild[45] that the luminosity coefficients of Wright's primaries be found by comparison with three filters, a red, a green, and a blue, whose relative brightnesses might then be evaluated from their spectral transmissions and the standard visibility function. This method would overcome the difficulties of heterochromatic photometry experienced by Wright in his attempt to measure the relative luminosities of his primaries by direct comparison. It is to be noted, however, that the trial and error method which we have used (see appendix I) accomplishes this evaluation without any auxiliary photometric measurements at all. Use has been made of the fact that equal quantities of the primaries are required to match the N. P. L. standard white; this fact, combined with the assumption that some certain visibility function, V, adequately represents Wright's observers, suffices to define the relative luminosities of the primaries. In the method suggested by Guild most of the uncertainty is removed by an appeal to the standard visibility function; in our method the assumed visibility function is made to bear the whole burden.

After the completion of the present paper another contribution by Wright[46] has become available. In this paper Wright makes use of much the same method developed independently in the present paper and he calculates a set of curves embodying the standard visibility function which in our notation would be $\rho_{w_s''wnn}$, $\gamma_{w_s''wnn}$, $\beta_{w_s''wnn}$. This set of distribution curves was not computed in the present work, but the values which he obtains agree closely with $\rho_{w_s'wnn}$, $\gamma_{w_s'wnn}$, $\beta_{w_s'wnn}$, which were found (step 1) incidental to the computation $\rho_{w_s'oaa}$, $\gamma_{w_s'oaa}$, $\beta_{w_s'oaa}$ (Table 5), the small differences being merely those to be expected from the slight difference in basic stimulus in the two cases, from rejection error in computation, and from errors in interpolating and extrapolating the mixture data.

In the discussion of this paper Guild (p. 213) admits the soundness

[44] K. S. Gibson and E. P. T. Tyndall, *Visibility of Radiant Energy*, B. S. Sci. Paper No. 475, pp. 158, 174; 1923.

[45] Discussion of Wright's paper, Trans. Opt. Soc., *30*, 161; 1928–29.

[46] W. D. Wright, *A Re-Determination of the Mixture Curves of the Spectrum*, Trans. Opt. Soc., *31*, 201–218; 1929–30.

of Wright's method of computation (which is, in effect, the same as that used in the present paper) provided the data be "*absolutely* accurate and *absolutely* consistent with the assumed normal visibility curve." Guild, however, states his belief that the method will not result in the best values in practical cases; and he supports this conviction by pointing out that subliminal errors in experimentally determined values of the trilinear coordinates of the spectrum may cause errors in computed values of the luminosity coefficients of the primaries which are greatly in excess of any reasonable tolerances. Wright agrees that Guild's point is well taken.

It should first be remarked that for the main purpose of the present paper the large variation in L_b with subliminal changes in R, G, B does not matter because we are interested to compare Wright's results (errors and all) with the O. S. A. "excitation" curves.

If, however, it be desired to deduce from Wright's results the most representative set of data, it is natural to assume that the standard visibility function adequately represents Wright's ten obervers. This assumption yields the data of Table 5. It is true that this assumption may lead into error, but this error is not to be corrected by obtaining the luminosity coefficients according to the method suggested by Guild. One way to avoid this error is to determine experimentally a visibility function adequately representing Wright's ten obervers and correctly adjusted to the macular pigmentation of his 36 observers; but this visibility function would differ from the standard visibility function, and the perfectly consistent data for these groups of observers would be judged to be non-representative of average, normal vision to the extent of the differences from the normal visibility function. Analogous procedures suggested by Guild in his discussion of Wright's paper seem open to the same objection. Hence, it would appear that the method of deducing distribution curves worked out independently by Wright and by the author is more likely to yield representative results. The large variation in L_b with subliminal change in R, G, B seems not to be due to this particular method of reducing data but is inherent in the visual mechanism, itself; it may be described as an indirect indication of the well-known fact that the admixture of a blue component of small luminosity may change the chromaticity of the mixture quite markedly.

A GENERAL FORMULA FOR THE COMPUTATION OF COLORIMETRIC PURITY*

By Deane B. Judd

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Abstract

The formula for purity is called general because if the trilinear coordinates of any color referred to any given set of primary color processes and any basic stimulus be given, the colorimetric purity of its stimulus may be computed with respect to any heterogeneous stimulus as the fixed stimulus of homo-heterogeneous analysis. The derivation of the formula is given and five special cases of historical and practical interest are worked out in order to demonstrate that previous formulas for the computation of colorimetric purity and an allied ratio are special cases of this one. The connection is also given between colorimetric purity and the allied ratio variously known as saturation, saturation fraction, and Sättigung.

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I. Introduction

The definition of colorimetric purity in American literature and the methods of computing it have been fairly uniform, perhaps due to the comparative recency of its introduction in this country. Abroad, however, a number of allied concepts are variously known by this name, and this concept given, in addition, various other names. Discussion of these concepts, and indeed attempts to provide nomenclature adequate for discussion, have been handicapped by lack of more than fragmentary knowledge of the properties of the concepts, their interrelations and their dependence on choice of coordinate system. It is the purpose of the present paper to supply this lack by deriving a general formula which embraces all these allied concepts in a general system of coordinates.

II. The Problem

The tristimulus mode of specifying colors has long been recognized as the fundamental one. According to this mode a color is specified by giving the amounts of three stimuli whose mixture is requisite to produce it. The particular three stimuli in terms of which other stimuli are expressed according to the colors they evoke are called primary stimuli

* Publication approved by the Director of the Bureau of Standards of the U.S. Department of Commerce.

and the colors evoked by them are called primary colors. The choice of primary stimuli is virtually unlimited; for example, many groups of three spectrum stimuli may be chosen, or stimuli produced by means of source-and-filter combinations may be substituted for any or all of the spectrum stimuli. The change from one set of primaries to another is merely equivalent to a change in coordinate system. The formula for colorimetric purity to be derived holds for any choice of primaries; that is, the transformation from one coordinate system to another is implied in the formula itself.

Let ρ, γ, β be the O. S. A. "excitation" curves. These functions of wave length specify the spectrum colors in terms of a certain choice of primaries according to an hypothetical observer whose characteristics are known to approximate satisfactorily those of average, normal vision.[1] The O. S. A. "excitation" curves are distinguished by their simplicity from the infinite number of other sets of curves which may represent the characteristics of the same hypothetical observer; that is, none of the ordinates of these distribution curves is less than zero, and many are zero. Any one of these infinite number of other sets of curves, say χ, ψ, ζ, may be computed from ρ, γ, β as follows:

$$\left.\begin{aligned} \chi &\equiv K_1\rho + K_2\gamma + K_3\beta \\ \psi &\equiv K_4\rho + K_5\gamma + K_6\beta \\ \zeta &\equiv K_7\rho + K_8\gamma + K_9\beta \end{aligned}\right\} \tag{1}$$

where K_1 to K_9 are arbitrary constants such that:

$$\begin{vmatrix} K_1 & K_2 & K_3 \\ K_4 & K_5 & K_6 \\ K_7 & K_8 & K_9 \end{vmatrix} \neq 0. \tag{2}$$

Define:

$$\left.\begin{aligned} \bar{\chi} &\equiv \int_0^\infty \chi ET d\lambda \\ \bar{\psi} &\equiv \int_0^\infty \psi ET d\lambda \\ \bar{\zeta} &\equiv \int_0^\infty \zeta ET d\lambda \end{aligned}\right\} \tag{3}$$

[1] For the sake of being specific we take the O. S. A. "excitation" curves in the form extrapolated by Priest and Gibson (J.O.S.A. & R.S.I., *10*, 230; 1925); but we might nearly as well have taken the O. S. A. "excitation" curves in their original form (J.O.S.A. & R.S.I., *6*, 548; 1922) or some other set of curves which yield a satisfactory approximation to average, normal vision.

where E is the spectral distribution of energy of the source used and T is the spectral transmission (or spectral reflectance in case of a reflecting surface) of the sample which the source illuminates. The product, ET, as a function of wave length is the spectral distribution of energy, therefore, of the stimulus whose colorimetric purity is to be computed[2] by recourse to the mixture data of ρ, γ, β. The numbers, $\bar{\chi}, \bar{\psi}, \bar{\zeta}$, may be used to specify the color evoked by the stimulus specified by ET because all stimuli for which $\bar{\chi}, \bar{\psi}, \bar{\zeta}$ are identical produce identical colors within the visual mechanism specified by ρ, γ, β.[3]

Define:

$$\left.\begin{aligned} X &\equiv \chi/(\chi+\psi+\zeta) \\ Y &\equiv \psi/(\chi+\psi+\zeta) \\ Z &\equiv \zeta/(\chi+\psi+\zeta) \end{aligned}\right\}. \tag{4}$$

X, Y, Z are called the trilinear coordinates of the spectrum; they refer to some set of three primary processes and some basic stimulus[4] which may be inferred from the values of the constants, K_1 to K_9, of relations (1) and (2).

Define:

$$\left.\begin{aligned} x &\equiv \bar{\chi}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \\ y &\equiv \bar{\psi}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \\ z &\equiv \bar{\zeta}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \end{aligned}\right\}. \tag{5}$$

The numbers, x, y, z, are called the trilinear coordinates of the color evoked by the stimulus of spectral energy distribution, ET; these trilinear coordinates are referred to the same primary color processes and basic stimulus as X, Y, Z of relation (4).

When the energy of a stimulus is confined to a range of wave lengths,

[2] If the stimulus whose colorimetric purity is to be computed were radiant energy direct from a light source, a single symbol would naturally be used for ET. Since most stimuli whose purities are to be computed are reflected or transmitted energies, it is convenient to use the product, ET.

[3] D. B. Judd, *Reduction of Data on Mixture of Color Stimuli*, B. S. Jour. Research, *4*, 515–548; 1930.

[4] The basic stimulus as defined by Priest is any stimulus whose color is represented at the center of the Maxwell triangle; that is, any stimulus whose color has the trilinear coordinates (see relation (5)), $x=y=z=1/3$. From relation (3) we see that the basic stimulus is any stimulus whose spectral energy distribution, E_b, satisfies the condition: $\int_0^\infty \chi E_b d\lambda = \int_0^\infty \psi E_b d\lambda = \int_0^\infty \zeta E_b d\lambda$. To specify a basic stimulus is a convenient way to specify the relative scale magnitudes of χ, ψ, ζ; these magnitudes could be just as rigorously specified by giving the spectral energy distribution, E_z, for which the numbers: $\int_0^\infty \chi E_z d\lambda$, $\int_0^\infty \psi E_z d\lambda$, $\int_0^\infty \zeta E_z d\lambda$, have any given relative values.

$\lambda-\Delta\lambda$ to $\lambda+\Delta\lambda$, so restricted that reduction of $\Delta\lambda$ makes no difference in the chromaticity of the evoked color, this stimulus is called homogeneous radiant energy of wave length, λ. Homogeneous radiant energy, E_λ, may be colorimetrically specified from relation (3) by the definite integrals:

$$\int_{\lambda-\Delta\lambda}^{\lambda+\Delta\lambda} \chi E_\lambda d\lambda, \quad \int_{\lambda-\Delta\lambda}^{\lambda+\Delta\lambda} \psi E_\lambda d\lambda, \quad \int_{\lambda-\Delta\lambda}^{\lambda+\Delta\lambda} \zeta E_\lambda d\lambda,$$

which for brevity are written hereafter as: $\bar{\chi}_\lambda$, $\bar{\psi}_\lambda$, $\bar{\zeta}_\lambda$. Note that these three values are proportional to: χ, ψ, ζ, for the wave length, λ.

Heterogeneous radiant energy of spectral distribution, E_w, is specified colorimetrically from relation (3) by the definite integrals:

$$\int_0^\infty \chi E_w d\lambda, \quad \int_0^\infty \psi E_w d\lambda, \quad \int_0^\infty \zeta E_w d\lambda,$$

which are abbreviated hereafter as: $\bar{\chi}_w$, $\bar{\psi}_w$, $\bar{\zeta}_w$. The subscript, w, is chosen because it suggests that the heterogeneous radiant energy is, as is usual, such as to evoke a "white" or "neutral" color.

The mixture of homogeneous radiant energy, E_λ, with heterogeneous radiant energy, E_w, is specified colorimetrically by the three sums of these definite integrals:

$$\bar{\chi}_\lambda + \bar{\chi}_w, \quad \bar{\psi}_\lambda + \bar{\psi}_w, \quad \bar{\zeta}_\lambda + \bar{\zeta}_w,$$

that is, for the mixture:

$$\left.\begin{aligned} \bar{\chi} &= \bar{\chi}_\lambda + \bar{\chi}_w \\ \bar{\psi} &= \bar{\psi}_\lambda + \bar{\psi}_w \\ \bar{\zeta} &= \bar{\zeta}_\lambda + \bar{\zeta}_w \end{aligned}\right\}. \tag{6}$$

Hence, from relation (5), the trilinear coordinates of the color evoked by this mixture are:

$$\left.\begin{aligned} x &= \frac{\bar{\chi}_\lambda + \bar{\chi}_w}{\bar{\chi}_\lambda + \bar{\psi}_\lambda + \bar{\zeta}_\lambda + \bar{\chi}_w + \bar{\psi}_w + \bar{\zeta}_w} \\ y &= \frac{\bar{\psi}_\lambda + \bar{\psi}_w}{\bar{\chi}_\lambda + \bar{\psi}_\lambda + \bar{\zeta}_\lambda + \bar{\chi}_w + \bar{\psi}_w + \bar{\zeta}_w} \\ z &= \frac{\bar{\zeta}_\lambda + \bar{\zeta}_w}{\bar{\chi}_\lambda + \bar{\psi}_\lambda + \bar{\zeta}_\lambda + \bar{\chi}_w + \bar{\psi}_w + \bar{\zeta}_w} \end{aligned}\right\} \tag{7}$$

The trilinear coordinates, x_w, y_w, z_w, of the color evoked by the heterogeneous component of the mixture are:

$$\left.\begin{aligned} x_w &= \bar{\chi}_w/(\bar{\chi}_w + \bar{\psi}_w + \bar{\zeta}_w) \\ y_w &= \bar{\psi}_w/(\bar{\chi}_w + \bar{\psi}_w + \bar{\zeta}_w) \\ z_w &= \bar{\zeta}_w/(\bar{\chi}_w + \bar{\psi}_w + \bar{\zeta}_w) \end{aligned}\right\}. \tag{8}$$

Define:

$$L_\lambda \equiv \bar{\rho}_\lambda L_r + \bar{\gamma}_\lambda L_g + \bar{\beta}_\lambda L_b \tag{9}$$

where L_r, L_g, L_b are any constants not all zero. $\bar{\rho}_\lambda$, $\bar{\gamma}_\lambda$, $\bar{\beta}_\lambda$ represent definite integrals analogous to those represented by $\bar{\chi}_\lambda$, $\bar{\psi}_\lambda$, $\bar{\zeta}_\lambda$. We use the symbol, L_λ, (luminosity) because when $L_r/(L_r+L_g+L_b)=0.45$, $L_g/(L_r+L_g+L_b)=0.54$, and $L_b/(L_r+L_g+L_b)=0.01$, or values not greatly differing from these, L_λ varies according to wave length very closely as the luminosity of the equal-energy spectrum determined experimentally.[5] We refrain from calling L_λ the luminosity because L_λ deserves such a name only for the special case just mentioned. Although this special case commands the major interest because of the possibility of determining it directly by experiment[6] some attention has been given another case, $L_r=L_g=L_b$, as will appear later.

It has been shown that:[7]

$$L_\lambda = \bar{\chi}_\lambda L_x + \bar{\psi}_\lambda L_y + \bar{\zeta}_\lambda L_z \tag{9a}$$

where:

$$\left.\begin{aligned} L_x &= \frac{1}{\Delta}\left[L_r\begin{vmatrix} K_5 & K_8 \\ K_6 & K_9 \end{vmatrix} + L_g\begin{vmatrix} K_7 & K_4 \\ K_9 & K_6 \end{vmatrix} + L_b\begin{vmatrix} K_4 & K_7 \\ K_5 & K_8 \end{vmatrix}\right] \\ L_y &= \frac{1}{\Delta}\left[L_r\begin{vmatrix} K_8 & K_2 \\ K_9 & K_3 \end{vmatrix} + L_g\begin{vmatrix} K_1 & K_7 \\ K_3 & K_9 \end{vmatrix} + L_b\begin{vmatrix} K_7 & K_1 \\ K_8 & K_2 \end{vmatrix}\right] \\ L_z &= \frac{1}{\Delta}\left[L_r\begin{vmatrix} K_2 & K_5 \\ K_3 & K_6 \end{vmatrix} + L_g\begin{vmatrix} K_4 & K_1 \\ K_6 & K_3 \end{vmatrix} + L_b\begin{vmatrix} K_1 & K_4 \\ K_2 & K_5 \end{vmatrix}\right] \end{aligned}\right\} \tag{9b}$$

and where

$$\Delta = \begin{vmatrix} K_1 & K_2 & K_3 \\ K_4 & K_5 & K_6 \\ K_7 & K_8 & K_9 \end{vmatrix},$$

the constants, K_1 to K_9, being those of relations (1) and (2).

[5] D. B. Judd, *Chromatic Visibility Coefficients by the Method of Least Squares*, J.O.S.A. & R.S.I., *10*, 635–651; 1925.

[6] See footnote 9, p. 734, and footnote 11, p. 735.

[7] See footnote 3, p. 731.

Define:

$$L_m \equiv \bar{\chi}L_x + \bar{\psi}L_y + \bar{\zeta}L_z. \tag{10}$$

The subscript, m, may be read, "of the mixture," and the analogous subscript, h, of relations (9) and (9a) may be read, "of the homogeneous component."

The ratio, L_h/L_m, is of particular interest in colorimetry because in the special case[8] where it is a ratio of luminosities it may be determined experimentally[9] by the methods of heterochromatic photometry, and it has been given a special name, the colorimetric purity. We may now state the problem.

Given: (1) values of the trilinear coordinates, x, y, z, of a color produced by the mixture of an homogeneous and an heterogeneous component, (2) the trilinear coordinates, x_w, y_w, z_w, of the color evoked by the heterogeneous component alone, (3) the trilinear coordinates, X, Y, Z, of the color evoked by the homogeneous component alone, (4) the constants, K_1 to K_9, specifying the primaries and basic stimulus of the distribution curves as related to the O. S. A. "excitation" curves, and (5) the constants, L_r, L_g, L_b; required: the ratio, L_h/L_m.

III. THE SOLUTION

The derivation of the general formula has been foreshadowed by the derivation of a special case of it.[10] From relation (9a) and definition (10) we have:

$$\frac{L_h}{L_m} = \frac{\bar{\chi}_h L_x + \bar{\psi}_h L_y + \bar{\zeta}_h L_z}{\bar{\chi}L_x + \bar{\psi}L_y + \bar{\zeta}L_z}$$

which, in virtue of relation (6), becomes:

$$\frac{L_h}{L_m} = \frac{\bar{\chi}_h L_x + \bar{\psi}_h L_y + \bar{\zeta}_h L_z}{\chi_h L_x + \bar{\psi}_h L_y + \bar{\zeta}_h L_z + \chi_w L_x + \bar{\psi}_w L_y + \zeta_w L_z}.$$

After the factor, $\bar{\chi}_h + \bar{\psi}_h + \zeta_h$, abbreviated as σ, has been divided out, this ratio becomes, from definition (4):

$$\frac{L_h}{L_m} = \frac{XL_x + YL_y + ZL_z}{XL_x + YL_y + ZL_z + \frac{\bar{\chi}_w}{\sigma}L_x + \frac{\bar{\psi}_w}{\sigma}L_y + \frac{\bar{\zeta}_w}{\sigma}L_z}. \tag{11}$$

[8] See relation (9) et seq.

[9] I. G. Priest, *Apparatus for the Determination of Color in Terms of Dominant Wave-Length, Purity and Brightness*, J.O.S.A. & R.S.I., *8*, 173–200; 1924.

[10] D. B. Judd, *The Computation of Colorimetric Purity*, J.O.S.A. & R.S.I., *13*, 136–138; 1926.

From definition (4) and relations (7) and (8) it follows at once that:

$$\bar{\chi}_w/\sigma = x_w(x - X)/(x_w - x)$$
$$\bar{\psi}_w/\sigma = y_w(y - Y)/(y_w - y)$$
$$\bar{\zeta}_w/\sigma = z_w(z - Z)/(z_w - z).$$

By substituting these three values in relation (11) a solution of the problem is obtained; but this solution may be expressed more simply by taking account of the collinearity condition:[11]

$$\frac{x - X}{x_w - x} = \frac{y - Y}{y_w - y} = \frac{z - Z}{z_w - z} \tag{13}$$

which also follows from (4), (7) and (8); it states that the points (X, Y, Z), (x, y, z) and (x_w, y_w, z_w) lie on the same straight line. In virtue of relation (13), relation (11) may be written:

$$\frac{L_h}{L_m} = \frac{XL_x + YL_y + ZL_z}{XL_x + YL_y + ZL_z + F(x_wL_x + y_wL_y + z_wL_z)} \tag{14}$$

where $F = (x-X)/(x_w-x) = (y-Y)/(y_w-y) = (z-Z)/(z_w-z)$. Relation (14) is a solution of the problem since X, Y, Z, x, y, z, x_w, y_w, z_w are known and L_x, L_y, L_z may be found from relation (9b) since the constants, K_1 to K_9, and L_r, L_g, L_b are given.

IV. FIVE SPECIAL CASES

1. For routine computation of colorimetric purity at the National Bureau of Standards it has been customary to choose constants, K_1 to K_9, all zero except K_1, K_5 and K_9 (which retains all the simplicity of the O. S. A. "excitation" curves) and so to adjust K_1, K_5 and K_9 that the heterogeneous stimulus is represented at the center of the Maxwell triangle, that is, the heterogeneous stimulus is chosen as the basic stimulus;[12] hence, the constants are taken so that:

$$\int_0^\infty \chi E_w d\lambda = \int_0^\infty \psi E_w d\lambda = \int_0^\infty \zeta E_w d\lambda$$

which, from this special case of relation (1), may be written:

$$K_1 \int_0^\infty \rho E_w d\lambda = K_5 \int_0^\infty \gamma E_w d\lambda = K_9 \int_0^\infty \beta E_w d\lambda.$$

[11] I. G. Priest, *The Computation of Colorimetric Purity*, J.O.S.A. & R.S.I., *9*, 520; 1924. D. B. Judd, *The Computation of Colorimetric Purity*, J.O.S.A. & R.S.I., *13*, 138; 1926.

[12] See footnote 4, p. 731.

L_x, L_y and L_z are chosen so that L_h of relation (9a) may be taken to represent the luminosity of the equal-energy spectrum;[13] hence we may write colorimetric purity, p, instead of L_h/L_m in relation (14) which becomes; since $x_w=y_w=z_w=1/3$:

$$p=\frac{XL_x+YL_y+ZL_z}{XL_x+YL_y+ZL_z+F(L_x+L_y+L_z)/3}.$$

Since, as is permissible (see relation (9) et seq), it is customary to set $L_x+L_y+L_z=1$, a further simplification results:

$$p=\frac{XL_x+YL_y+ZL_z}{XL_x+YL_y+ZL_z+F/3} \tag{15}$$

where $F/3=(x-X)/(1-3x)=(y-Y)/(1-3y)=(z-Z)/(1-3z)$. Relation (15) may be recognized as a restatement of the Judd R, Judd G and Judd B forms of purity formula[14] with obvious changes in symbology. It is also identical with other forms of purity formula previously derived (Ives, Priest, Tuckerman). A formula developed by Froelich[15] is identical with relation (15) providing L_h of relation (9a) be taken equal to her L_λ, and providing her ratio, α, determined graphically by measuring two distances on the mixture diagram be given its analytic equivalent, $1/(1+F)$. Two formulas developed by Guild[16] also involve ratios of lengths on the mixture diagram, the ratio in the first formula given (Guild's p. 161) being equal to $1/(1+F)$ and that in the second formula (p. 162) being equal to $1/F$. When these substitutions are made, the first formula can readily be shown to be identical with the Ives form which has already been proven identical with relation (15); the second formula becomes relation (15) directly.

It may be remarked that the various forms of purity formula with this convention of making the heterogeneous stimulus the basic stimulus of the system arise chiefly because (1) the coordinates of any pair of the three collinear points may be taken to form the luminosity sums, as, for example, in relation (15) which takes the coordinates of the spectrum point and the "white" point; (2) the ratio of lengths may be ex-

[13] They may be found from relation (9b) if L_r, L_g and L_b are appropriately chosen (see relation (9) et seq) or they may be found de novo.

[14] See footnote 10, p. 734.

[15] Clara L. Froelich, *Algebraic Methods for the Calculation of Color Mixture Transformation Diagrams*, J.O.S.A. & R.S.I., *9*, 39; 1924.

[16] J. Guild, *The Geometrical Solution of Colour Mixture Problems*, Trans. Opt. Soc., *26*, 161–162; 1924–25.

pressed as the ratio of the segments or as the ratio of either segment to the total, as, for example, in relation (15) which uses F, the ratio of the segments; and (3) this ratio of lengths may be evaluated from any parallel projection of the line, as, for example, in relation (15) which evaluates the ratio, F, from projections onto the axes of a mixture diagram in rectangular coordinates or onto the bisectors of the angles of the equilateral triangle, or as, for another example, in Ives' formulas which evaluate a ratio of lengths from projections onto the sides of the equilateral triangle, or as in the formulas of Froelich and Guild which evaluate the ratios directly from the lengths, themselves. The various forms of purity formula are all useful, the particular form of purity formula to be chosen depending on the nature of the problem.

2. Another choice of primary processes which tends more toward convenience for those who think in terms of the opponent-colors theory of vision (Hering) is that which takes "neutral" as one of the three primary processes. The chromatic aspect of the response is then described solely by the other two components which have negative as well as positive values,—such as a red process whose negative is green, and a yellow process whose negative is blue. For such a choice, if further the heterogeneous stimulus be such as to evoke the "neutral" color, and if the "neutral" color be chosen as the third primary whose distribution in the spectrum is given by ζ, the constants, K_1 to K_9, of relation (1) must satisfy the condition:

$$\int_0^\infty \chi E_w d\lambda = \int_0^\infty \psi E_w d\lambda = 0$$

which from relation (1) is equivalent to:

$$K_1 \int_0^\infty \rho E_w d\lambda + K_2 \int_0^\infty \gamma E_w d\lambda + K_3 \int_0^\infty \beta E_w d\lambda = 0$$

$$K_4 \int_0^\infty \rho E_w d\lambda + K_5 \int_0^\infty \gamma E_w d\lambda + K_6 \int_0^\infty \beta E_w d\lambda = 0.$$

From relation (8) and the definitions of $\bar{\chi}_w$, $\bar{\psi}_w$, $\bar{\zeta}_w$, it follows that $x_w = y_w = 0$, and $z_w = 1$. Relation (14) becomes:

$$\frac{L_h}{L_m} = \frac{XL_x + YL_y + ZL_z}{XL_x + YL_y + ZL_z + FL_z}$$

where

$$F = (x - X)/(- x) = (y - Y)/(- y) = (z - Z)/(1 - z).$$

A further simplification is possible if K_7, K_8 and K_9 be chosen equal respectively to L_r, L_g and L_b. From relation (9b) it follows that $L_x = L_y = 0$. And if still further we arbitrarily set $L_z = 1$, as is permissible, we may write:

$$L_h/L_m = Z/(Z + F).$$

If, further, L_r, L_g and L_b be chosen so that the ratio may be considered a luminosity ratio as in case (1), this relation may be written as a very simple formula for colorimetric purity:

$$p = (1 - 1/z)/(1 - 1/Z). \tag{17}$$

Since Schrödinger's "vierfarbentheorie" curves[17] satisfy all the conditions under which relation (17) was derived, this relation may be taken for colorimetric purity if the trilinear coordinates X, Y, Z, and x, y, z, are computed from those curves. The simplicity of the formula raises the question whether Schrödinger's choice of primary color processes, which were chosen because of their interest for psycho-physiological speculation, might not be a favorable choice purely for the purpose of simplifying the computation of colorimetric purity. This question is discussed presently.

3. Some attention has been given to the ratio, L_h/L_m in the special case for which $L_x = L_y = L_z$,[18] perhaps chiefly because of the relative ease with which it may be computed. Since, from relation (4), $X+Y+Z=1$, relation (14) for this special case may be written:

$$L_h/L_m = 1/(1 + F)$$

or:

$$L_h/L_m = (x_w - x)/(x_w - X) = (y_w - y)/(y_w - Y) = (z_w - z)/(z_w - Z). \tag{18}$$

Inspection of relation (18) shows that L_h/L_m for this case is the distance on the mixture diagram from the point, (x,y,z), to the "white" point, (x_w, y_w, z_w), divided by the distance from the "spectrum" point, (X,Y,Z), to the "white" point; hence the ratio may in this case be determined quite easily from the mixture diagram directly.

It should be noted, however, that L_h/L_m in this case is not necessarily

[17] E. Schrödinger, *Über das Verhältnis der Vierfarben- zur Dreifarbentheorie*, Sitz. Akad. Wiss., Wien, IIa, *134*, 479; 1925.

[18] W. Dziobek and M. Pirani, *Normung von Signalgläsern*, Proc. International Congress on Illumination, Saranac Inn, 818–833; 1928. I. Runge, *Die Einheitsmengen im Maxwell-Helmholtzschen Farbdreieck und die Bestimmung der Farbsättigung*, ZS. f. Instrumentenk. *49*, 600–603; 1929. S. Rösch, *Darstellung der Farbenlehre für die Zwecke des Mineralogen*, Fortschritte d. Mineralogie, Kristallographie u. Petrographie, *30*, 135–136; 1929.

the colorimetric purity because L_x, L_y and L_z have not been especially chosen to make L_λ of relation (9a) proportional to the luminosity of the equal-energy spectrum. It should also be noted that this special case loses the characteristic which is the probable cause of its existence if a different choice of primaries or basic stimulus be made; that is, with certain trivial exceptions, the coefficients, L_x, L_y, L_z, in another system of coordinates would no longer be equal; hence, this special case of the ratio L_λ/L_m would be just as hard to compute as colorimetric purity or any other special case of relation (14).[19] The condition for the exceptions may be seen by inspection of relation (9b).

4. In order to facilitate the computation of colorimetric purity, Runge[20] has suggested another choice of constants, K_1 to K_9. For this choice the simple formula (18) just derived results in the special case of L_λ/L_m which has been defined as the colorimetric purity. It is interesting to see how the choice of constants, K_1 to K_9, suggested by Runge insures that relation (18) results in colorimetric purity.

Take L_r, L_g, L_b such that L_λ of relation (9) gives the distribution of luminosity in the equal-energy spectrum. This choice permits us to substitute p for L_λ/L_m. Then the constants, K_1 to K_9, are chosen all zero except K_1, K_5, K_9 which are taken as L_r, L_g, L_b, respectively. From relation (9b) it will be discovered that, for this choice: $L_x=L_y=L_z=1$; hence it follows that relation (18) in this special case gives colorimetric purity.

According to this plan, however, the basic stimulus of the $\rho\gamma\beta$ system is taken as the heterogeneous stimulus in the $\chi\psi\zeta$ system, necessitating that:

$$\int_0^\infty \rho E_w d\lambda = \int_0^\infty \gamma E_w d\lambda = \int_0^\infty \beta E_w d\lambda$$

which, from relation (1), is equivalent to:

$$(1/L_r)\int_0^\infty \chi E_w d\lambda = (1/L_g)\int_0^\infty \psi E_w d\lambda = (1/L_b)\int_0^\infty \zeta E_w d\lambda$$

whence, from relation (8), it is evident that:

$$x_w = L_r/(L_r + L_g + L_b)$$
$$y_w = L_g/(L_r + L_g + L_b)$$
$$z_w = L_b/(L_r + L_g + L_b).$$

[19] Another way of making the same statement is that if L_x, L_y and L_z in another system be made equal, in general, the number defined by such a choice will not be identical with L_λ/L_m in this special case.

[20] See footnote 18, p. 738.

Since L_b in this case is about 1/50 of either L_r or L_g, it is evident that the "white" point, (x_w, y_w, z_w), falls relatively near one side of the Maxwell triangle.[21] This consequence of Runge's choice of coordinate system is emphasized by setting, as is usual, $L_r+L_g+L_b=1$, which permits relation (18) for this special case to be written:

$$p = (L_r - x)/(L_r - X) = (L_g - y)/(L_g - Y) = (L_b - z)/(L_b - Z). \quad (19)$$

5. Another coordinate system has since been proposed by Judd[22] to facilitate the computation of colorimetric purity. This system avoids the close approach of the spectrum locus to the "white" point.

Take $K_2=K_3=K_7=K_8=0$, $K_4=L_r$, $K_5=L_g$ and $K_6=L_b$, where L_r, L_g, L_b, as before, are chosen so that L_k of relation (9) gives the distribution of luminosity in the equal-energy spectrum. As in case (2) it is discovered from relation (9b) that two of the three numbers, L_x, L_y, L_z, are zero; that is, in this case: $L_x=L_z=0$, $L_y=1$.

K_1, K_9, and $L_r+L_g+L_b$ are chosen so as to make E_w the basic stimulus which, from relation (1), requires that:

$$K_1 \int_0^\infty \rho E_w d\lambda = K_9 \int_0^\infty \beta E_w d\lambda$$

$$= L_r \int_0^\infty \rho E_w d\lambda + L_g \int_0^\infty \gamma E_w d\lambda + L_b \int_0^\infty \beta E_w d\lambda.$$

Since, by this choice, $x_w=y_w=z_w=1/3$, relation (14) may be written, for $Y \neq 1/3$:

$$p = (3 - 1/y)/(3 - 1/Y). \quad (20)$$

An equally simple form may be written for $Y=1/3$.[23]

Relation (17) is just as simple for routine computation as relation (20) provided the trilinear coordinates (x,y,z) of the mixture are at hand. The distribution curves leading up to relation (17), however, are less simple for routine computation of the coordinates, x, y, z, than those leading up to relation (20) because they embody more ordinates different from zero and because they embody negative as well as positive ordinates; hence relation (20), considering the entire computation of colorimetric purity from spectral distribution of energy, is more advantageous than relation (17).

[21] See Runge's Fig. 2 (footnote 18, p. 738).
[22] See footnote 3, p. 731.
[23] See footnote 3, p. 731.

Relation (20) is not quite so convenient for routine computation as relation (19) since it involves the finding of two reciprocals in addition to the two subtractions and one division indicated in relation (19). However, the ease with which X, Y, Z may be found from x, y, z within the coordinate system of relation (20) is generally considerably greater than for the coordinate system of relation (19) because the coordinate system of relation (20) avoids the close approach to the spectrum locus of the "white" point, (x_w, y_w, z_w).[24]

If any coordinate system is to be chosen for the specific purpose of facilitating the computation of colorimetric purity, it would seem that the adoption of that leading to relation (20) would be the most advantageous in routine computation.

V. PURITY AND SATURATION

Discussion of the terms, purity and saturation, is facilitated by the introduction of a ratio, L_a/L_m, which is in all respects save one the same as L_h/L_m. Since this ratio is of considerably less importance than L_h/L_m, there is little interest in showing how it may be computed from one set of primaries after having been defined in terms of another as was done with L_h/L_m. It suffices to write a relation analogous to relation (14) which will serve to define L_a/L_m, where the subscript, a, is to be read, "absolute," (or, "basic," according to Priest) and to be understood to refer to a color represented on the edge of the Maxwell triangle:

$$\frac{L_a}{L_m} \equiv \frac{X_aL_x + Y_aL_y + Z_aL_z}{X_aL_x + Y_aL_y + Z_aL_z + F_a(x_wL_x + y_wL_y + z_wL_z)} \tag{21}$$

where $F_a \equiv (x-X_a)/(x_w-x)$ or $(y-Y_a)/(y_w-y)$ or $(z-Z_a)/(z_w-z)$. Relation (21) differs from relation (14) only by the substitution of X_a, Y_a, Z_a respectively for X, Y, Z. Since L_h/L_m expresses by its approach to unity the degree of approach of the color, (x,y,z), to the color, (X, Y, Z), the ratio, L_a/L_m, similarly gives the degree of approach of the color, (x,y,z), to the color (X_a,Y_a,Z_a). The color, (X,Y,Z), used as a reference in the one case may be evoked by some portion of the spectrum; in the second case the color, (X_a,Y_a,Z_a), used as a reference is represented on an edge of the Maxwell triangle; we know, therefore, that one of X_a, Y_a, Z_a is zero.

We have seen that two special cases of the ratio, L_h/L_m, are of in-

[24] Compare Runge's Fig. 2 (see footnote 18, p. 738) with Judd's Fig. 5 (see footnote 3, p. 731).

terest, the one in which $L_x = L_y = L_z$, and the one in which L_x, L_y, L_z are chosen so that L_h of relation (9a) may be taken as the luminosity of the equal-energy spectrum. These two special cases also command all the attention given to the ratio, L_a/L_m.

The first case yields a formula analogous to relation (18) which becomes for $x_w = y_w = z_w = 1/3$: $L_a/L_m = (1-3x)/(1-3X_a) = (1-3y)/(1-3Y_a) = (1-3z)/(1-3Z_a)$ which may be written:

$$L_a/L_m = 1 - 3c \tag{22}$$

where c is x or y or z according as X_a or Y_a or Z_a is zero. This ratio was defined by Exner[25] as "Sättigung," for the coordinate system based on the Grund-Empfindungs-Curven of König and Dieterici.[26] Exner's definition has been followed by a number of European workers. It has been recently restated as a "not wholly unnatural measure of saturation" by Schrödinger[27] who also refers to it as "Sättigung."

The second case (for $x_w = y_w = z_w = 1/3$) has been given by Martin[28] as one of three variables for specifying color stimuli; the ratio, L_a/L_m, is defined by a numerical example referring to Abney's "sensation" curves and is called the "saturation fraction." This second case was also mentioned later by Priest[29] incidental to the derivation of a formula for colorimetric purity; he named the ratio, L_a/L_m, for this case the "absolute purity." Still later this ratio was defined by Haschek[30] who called it "Sättigung."

Probably two reasons may account for choosing as a reference the edges of the Maxwell triangle as in the ratio, L_a/L_m. The first is merely that the resulting formula is simpler because one of X_a, Y_a, Z_a is always zero; this is a logical extension of the motive which probably led up to relation (18) and applies with particular force to the first case which results in the simple relation (22). The second reason is the suspicion, or the delusion, or, perhaps, the hope that the edges of the Maxwell triangle of the particular coordinate system chosen possess a fundamental theoretical significance by virtue of which they deserve the

[25] F. Exner, Sitz. Akad. Wiss., Wien, I, *119*, 233; 1910.

[26] A. König and C. Dieterici, *Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsvertheilung im Spectrum*, ZS. f. Psych. u. Physiol. d. Sinnesorgane, *4*, 241–347; 1893; or see A. König, Ges. Abh., Leipzig, Barth, 214–321; 1903.

[27] E. Schrödinger, Müller-Pouillets Lehrbuch d. Physik, 2nd Ed., *2*, 482–484; 1926.

[28] L. C. Martin, *Colour and Methods of Colour Reproduction*, London, Blackie, 133; 1923.

[29] I. G. Priest, *The Computation of Colorimetric Purity*, J.O.S.A. & R.S.I., *9*, 509; 1924.

[30] E. Haschek, *Quantitative Beziehungen in der Farbenlehre*, Sitz. Akad. Wiss., Wien, IIa, *136*, 461–468; 1927.

dignity of being chosen as reference lines. If the primary color processes of the coordinate system were known to be processes functionally distinct somewhere within the visual mechanism, then it could be argued that every color process represented by a point outside the Maxwell triangle would be non-existent, or imaginary. The edges of the triangle would then represent the colors which differ from an achromatic color of the same brilliance by the maximum amount possible, and hence they would form a rather natural reference locus. However, the argument by which the König Grund-Empfindungs-Curven are taken to give the distributions of the true primary processes is by no means complete. The primaries chosen by König satisfy two criteria (1) they are a particular "red," "green," and "blue" whose hues are close to three of the four psychologically unitary hues; that is, the hues are nearly Urrot, Urgrün and Urblau, and (2) the distribution curves for these three primary processes may be used not only to describe the mixture relations of normal vision when used all three together, but also, by disregarding the one or the other of the "red" and "green" curves, the two most common types of partial color-blindness may be described. Now it is not necessary to assume that one distribution curve must be disregarded to account for partial color-blindness; this is merely the simplest assumption.[31] Furthermore, the König Grund-Empfindungs-Curven are not the only set of curves which satisfy the two conditions given; they are merely the simplest set which satisfy them; that is, the Maxwell triangle which they yield approaches as closely as possible to the spectrum locus as is consistent with the criteria just mentioned. Other sets of distribution curves satisfying the two conditions satisfied in the simplest possible way by the König curves would yield Maxwell triangles whose sides departed to a greater or less extent from the spectrum locus. Now if it should be desired to choose the particular triangle of these possible triangles whose sides represent the colors of the maximum saturation possible, it would be found that so far little evidence has been recorded to guide the choice. It is rather certain, however, that any triangle which touches the spectrum locus is not the right choice because it is an accepted fact that the colors of the spectrum viewed by a chromatically rested eye are not as saturated as the colors of the spectrum viewed by an eye adapted to the complementary color. But in all the triangles used by those proposing the ratio, L_a/L_m, as of fundamental impor-

[31] For the general form of dichromasy, see H. v. Helmholtz, Physiol. Optik, 2nd Ed., Leipsig, Voss, 458–462; 1896.

tance, the spectrum locus does touch at least one side of the triangle. Hence, our knowledge of the true physiological primaries, though much too incomplete to determine them, is sufficient to make certain that the coordinate systems thus far used for the ratio, L_a/L_m, fail to give that ratio a single vestige of theoretical importance.

It may be concluded, then that the interest attached to the ratio, L_a/L_m, like that attached to the ratio, L_k/L_m, is a purely practical one; either ratio for any choice of L_x, L_y, L_z will serve as one of three variables by which color stimuli may be classified according to the colors they evoke in the visual mechanism defined by ρ, γ, β. The special case of L_k/L_m when L_x, L_y, L_z are so chosen that it is a ratio of luminosities is, of course, of considerably greater interest than any other special case of either L_k/L_m or L_a/L_m, because this ratio alone can be evaluated by direct measurement.[32] To this special case, therefore, it has seemed expedient to give the name, colorimetric purity, p; and since this ratio serves all the purposes served by any other special case of either L_k/L_m or L_a/L_m there would seem to be little justification for paying any further attention to them. When the computation of colorimetric purity was not well understood it was natural that allied entities whose derivation seemed simpler should spring up. It should be evident, however, that methods of computation of colorimetric purity are now available which compare so favorably in simplicity with any of those for any of the allied entities, that the argument of mere ease of computation is no longer applicable.

It has become customary in some quarters, however, (Runge, Rösch)[33] to speak of purity and saturation (Reinheit and Sättigung) interchangeably and to remark that purity (or saturation) has been computed in four different ways. If we adopt the view that at least these four special cases deserve attention, it would seem conducive to clarity in discussion to adopt separate names for them. Terms proposed by Priest[34] for these four special cases together with an indication of their derivation appear in Table 1. It will be noted that, according to this terminology, the most important of the concepts, colorimetric purity, would be renamed "spectral brightness purity." In spite of the fact that previous writers have frequently applied the name "saturation" ("Sättigung") to one or another of these concepts, the suggested terminology does not in-

[32] See footnote 9, p. 734.
[33] See footnote 18, p. 738.
[34] Letter to Dr. M. Pirani, August 21, 1929.

TABLE 1. *Terminology suggested by Priest.*

Quantity to be named:			
L_a/L_m for $L_x=L_y=L_z$	L_h/L_m	L_a/L_m for $\bar{x}_h L_x+\bar{y}_h L_y+\bar{z}_h L_z$ equal to the luminosity of the equal-energy spectrum	L_h/L_m
Description:			
Hueful fraction of excitation relative to the sides of the triangle	Hueful fraction of excitation relative to the spectrum	Hueful fraction of brightness relative to the sides of the triangle	Hueful fraction of brightness relative to the spectrum
Suggested term:			
Basic excitation purity	Spectral excitation purity	Basic brightness purity	Spectral brightness purity

clude that term. This conforms to the recommendations of the O. S. A. committee on colorimetry[35] which reserve the term, saturation, to apply to the response to a color stimulus rather than to the stimulus, itself. The concepts thus far dealt with may be taken as stimulus terms because they merely serve to group together stimuli which evoke identical colors regardless of the chromatic condition of the visual mechanism.

If, however, we could compute from the trilinear coordinates of a color a number, s, satisfying the following conditions:

(a) $s=o$ for an achromatic color,

(b) $s_1=s_2$ for two colors appearing to be equally saturated,

then it would seem reasonable to name this number the "saturation," in which case there would be at hand a quantitative means of evaluating the concept, "saturation," which is now only qualitatively defined. Now both the ratios, L_a/L_m and L_h/L_m, satisfy condition (a) providing the heterogeneous stimulus whose color is represented at the point, (x_w, y_w, z_w), be so chosen as to evoke an achromatic color; but neither L_a/L_m nor L_h/L_m has been shown to satisfy condition (b). In fact, the failure of one special case of one of these ratios, L_a/L_m, for $L_x=L_y=L_z$, (Exner's "Sättigung," or basic excitation purity referred to the König fundamentals as primaries) to satisfy condition (b) has been proven experimentally by Seitz.[36] Other special cases which have been proposed do

[35] L. T. Troland, Report of Committee on Colorimetry for 1920–21, J.O.S.A. & R.S.I., *6*, 531–538; 1922.

[36] W. Seitz, *Über die Definition der Sättigung einer Farbe nach Helmholtz und Exner und über das Ostwaldsche Farbensystem*, ZS. f. Sinnesphysiol., II, *54*, 146–158; 1922; or see Phys. ZS., *23*, 297–301; 1922.

not differ from this case in such a way as to repair this deficiency; and, indeed, we may conclude from Seitz' work that no special case of either ratio satisfies condition (b) at all satisfactorily; so neither of these ratios deserve the name, "saturation."

A number which may prove to satisfy fairly closely both conditions (a and b) is $\int_0^p (dE/dp)dp$, where p is the colorimetric purity[37] of the stimulus and dE/dp is the reciprocal of the least difference in purity perceptible as a function of purity when the change in purity is made at constant brightness along some specified path on the color triangle.[38] $\int_0^p (dE/dp)dp$ may be described as the number of least perceptible differences between a color whose stimulus is of purity, p, and the color of the same brilliance whose stimulus is of purity, $p=0$. Since the saturation of a color depends nearly if not quite as much on the momentary condition of the visual mechanism as upon the stimulus, itself, it is evident that we may expect the number, $\int_0^p (dE/dp)dp$, to satisfy conditions (a) and (b) only when the visual mechanism is in such a state that the stimulus for which $p=0$ evokes an achromatic color. Since the saturation of the color evoked by a stimulus is dependent on the brightness as well as the purity of the stimulus, it is further evident that we may hope for $\int_0^p (dE/dp)dp$ to satisfy condition (b) only for the brightness to which dE/dp applies. Whether with these restrictions this number really may be made to satisfy conditions (a) and (b) and deserves the name, "saturation," is a matter which is plainly of considerable complexity; it has not yet been decided. An extensive quantity of experimental data has yet to be amassed before this definition of "saturation" may be more than tentatively accepted.

Before this definition, if acceptable, can be of convenient application, a way must be found of computing dE/dp from the trilinear coordinates of the color whose stimulus has a purity, p; but at least a

[37] We define this number in terms of p, but the definition could be equally well given by substituting for p any special case of either L_a/L_m or L_b/L_m.

[38] For instance, at constant dominant wave-length as adopted in defining this number by Jones and Lowry (L. A. Jones and E. M. Lowry, *Retinal Sensibility to Saturation Differences*, J.O.S.A. & R.S.I., *13*, 25–34; 1926) who identified it with "saturation," or at constant hue, or along such a path that this number is a minimum, as adopted by Schrödinger in defining "Sättigung" (E. Schrödinger, Müller-Pouillets Lehrb. d. Physik, 2nd Ed., *2*, 555–558; 1926). Schrödinger makes the not implausible assumption that this path coincides exactly with the path of constant hue. In general, there is quite a definite difference between the path of constant dominant wave-length and the path of constant hue, but we need not here choose between them for the purpose of this definition because there is no experimental evidence that the number defined on the one basis would differ at all importantly from that defined on the other.

start has already been made in this direction. Schrödinger[39] has derived an expression for dE/dp from theoretical ground; but this expression apparently has not yet been checked against experimental results. Judd[40] has discovered empirically an expression which agrees fairly well with such experimental results as are available, but the agreement is not sufficiently striking to lead to the belief that the expression is of more than temporary theoretical interest. It is evident, therefore, that our knowledge of vision still falls considerably short of a general quantitative evaluation of the response entity, saturation.

[39] E. Schrödinger, Grundlinien einer Theorie der Farbenmetrik im Tagessehen, Ann. d. Physik, (4), *63*, 483–520; 1920; see also footnote 38, p. 746.

[40] D B. Judd, *Purity and Saturation, A Saturation Scale for Yellow*, J.O.S.A. & R.S.I., *14*, 470; 1927; *Saturation of Colors Determined from the Visual Response Functions*, J.O.S.A. & R.S.I. *16*, 115; 1928.

ROCHESTER SECTION OF THE OPTICAL SOCIETY OF AMERICA

The Rochester Section of the Optical Society now comprises a membership of about 140. Officers for 1931–32 are as follows.

President	A. L. Schoen
Vice-president	C. E. Bahn
Secretary	R. Kingslake
Treasurer	M. E. Russell
Councillors at Large	F. M. Bishop, B. O'Brien, H. F. Kurtz, F. C. Fairbanks
Committees: Membership:	The Council
Entertainment:	F. E. Altman, C. E. Bahn, F. M. Bishop, R. Kingslake, M. E. Russell
Publicity:	T. R. Wilkins
Program:	T. R. Wilkins

The program for scheduled meetings during the present season is given below.

OCTOBER 13, 1931

1. Inspection of New Research Laboratories, Kodak Park.
   Welcome by Dr. C. E. K. Mees.

OCTOBER 22–24, 1931

2. Annual Meeting of the Optical Society of America, at Rochester.

NOVEMBER 3, 1931

3. Mr. J. C. Kurtz, Bausch and Lomb Optical Company.
   Subject:—"Optical Glass and its Manufacture."

NOVEMBER 17, 1931

4. Dr. S. L. Warren, Strong Memorial Hospital.
   Subject:—"The Production of Artificial Fever by Radio Waves."

DECEMBER 1, 1931

5. Mr. G. F. Metcalf, General Electric Company, Schenectady.
   Subject:—"New Developments in Vacuum Tubes for Optical Applications."

DECEMBER 15, 1931

6. Mr. C. B. Neblette, Eastman Kodak Company.
   Subject:—"The Microscope in the Service of Science and Industry."

JANUARY 12, 1932

7. Dr. Walter Clark, Eastman Kodak Company.
   Subject:—"Photographic Research Laboratories of Europe and Their Results."

JANUARY 26, 1932

8. Dr. H. Rosenberger, Rockefeller Institute for Medical Research.
   Subject:—"Some New Developments in Microcinematography."

FEBRUARY 9, 1932

9. Dr. Scott W. Smith, University of Rochester.
   Subject:—"Optical Researches of Michael Faraday."

FEBRUARY 23, 1932

10. To Be Announced.

MARCH 8, 1932

11. Dr. Brian O'Brien, University of Rochester.
Subject:—"The Application of Solar Radiation Measurements to Studies of the Upper Atmosphere."

MARCH 22, 1932

12. Dr. G. R. Harrison, Massachusetts Institute of Technology.
Subject:—"Photographic Photometry in the Schumann Region."

APRIL 5, 1932

13. Dr. F. K. Richtmyer, Cornell University.
Subject:—"Extension of X-ray Measurements Towards the Ultraviolet."

APRIL 19, 1932

14. Dr. H. A. Clark, Taylor Instrument Company.
Subject:—"Misleading Results from the Use of Temperature Measuring Devices."

MAY 3, 1932

15. Business Meeting.
Election of Officers.
Annual Dinner.

The following is an alphabetical list of members. Preceding each name are letters showing company affiliations and other pertinent data.

LIST OF MEMBERS

CODE

B-L—Bausch and Lomb Optical Co.
EC—Eastman Kodak Co., Camera Works
ES—Eastman Kodak Co., State Street Office
EK—Eastman Kodak Co., Kodak Park
HE—Eastman Kodak Co., Hawk-eye Works
R—Rochester Residents
T—Taylor Instrument Co.
U—University of Rochester
Hon.—Honorary Membership

HE	Aklin, George
U	Alling, Dr. Harold L.
HE	Altman, F. E.
B-L	Anderson, Arthur W.
B-L	Anderson, C. Einar
R	Archey, R. B.
B-L	Bahn, Carl E.
B-L	Baird, Gordon
R	Barber, Dr. Frank
B-L	Bausch, Carl L.
B-L	Bausch, Edward
B-L	Bausch, William
B-L	Belt, Haller
ES	Bishop, Dr. F. M.
Hon.	Bissell, Elmer J.
B-L	Bissell, Everett S.
ES	Boon, J. L.
HE	Bornemann, Wm.
ES	Brehm, F. W.
R	Bridge, Dr. Ezra
B-L	Bugbee, L. Willis, Jr.
T	Burger, Alfred
EK	Burroughs, R. E.
B-L	Butterfield, John V.
EK	Capstaff, J. G.
B-L	Challis, Frank E.
ES	Chapman, Dr. A. K.
T	Clark, Dr. H. A.
EK	Clark, Dr. Walter
EK	Cleveland, H. W.
B-L	Cook, Alan A.
EK	Crabtree, J. I.
EK	Crouch, Harold
EK	Davidson, S. W.
B-L	Davis, James M.
B-L	Drescher, Theodore B.
B-L	Drescher, William A. E.
HE	Eichelberger, Prof. W. S.
B-L	Eisenhart, M. H.
B-L	Ellestad, C. A.

U	Fairbanks, Prof. F. C.
ES	Fairbanks, H. N.
EK	Farley, Walter E.
U	Fassin, Gustave
EK	Fillius, Milton F.
B-L	Forrest, John W.
B-L	Foster, Leon V.
HE	Frederick, Dr. C. W.
ES	Fritts, Dr. Edwin C.
B-L	Gallasch, George
U	Gavett, Prof. Joseph W.
B-L	Graeper, W. W.
EK	Gray, Harry LeB.
EC	Green, Newton B.
HE	Guilford, R. S.
EK	Hall, Vincent C.
B-L	Hallauer, Carl S.
B-L	Hart, Edwin
HE	Higgins, F. L.
EK	Holmes, F. M. E.
B-L	Hudson, Miss L. M.
EK	Hutchinson, C. F.
B-L	Jobe, Frederick W.
EK	Jones, L. A.
R	Kennedy, Dr. E. H.
R	Kingslake, Mrs. H. G.
U	Kingslake, Prof. R.
B-L	Kolb, F. P.
B-L	Kurtz, Henry F.
B-L	Kurtz, J. C., Jr.
EK	Lane, Gerould T.
B-L	Leffler, G. H.
HE	Leggatt, S. T.
B-L	Levin, Dr. A. A.
B-L	Lomb, Adolph
B-L	Lomb, Carl F.
EK	Loveland, R. T.
EK	Lowry, Earl M.
ES	Lyman, Donald F.
B-L	Martin, Allan F.
ES	Martin, F. C.
R	McFarlane, Mrs. Ellen L.
EK	McFarlane, J. W.
EC	McRae, D. Brent
EK	Mees, Dr. C. E. K.
EK	Miller, O. E.
B-L	Moody, G. P.
ES	Morrison, C. A.
B-L	Nitchie, C. C.
B-L	Nixon, I. L.
T	Noyes, Dr. Bradford
Hon.	Nutting, Dr. P. G.
U	O'Brien, Dr. Brian
EK	Ord, P. R.
B-L	Parenteau, Charles A.
B-L	Parks, Vernon L.
B-L	Patterson, W. L.
R	Peckham, Robert H.
U	Petry, Ernest
B-L	Poser, Dr. Max
R	Priest, Everett L.
B-L	Quin, Elmer G.
B-L	Rayton, W. B.
R	Renaud, Charles
EK	Richardson, Earl E.
ES	Rusin, L. E.
EK	Russell, Marion E.
EK	Sandvik, Dr. Otto
EK	Schoen, A. L.
HE	Schultheiss, R.
EK	Semann, Dr. H. E.
B-L	Shaw, Darwin S.
EC	Sherman, F. C.
U	Smith, Dr. Scott W.
R	Snell, Dr. A. C.
B-L	Sterling, Scott
B-L	Steward, Willard P.
B-L	Straat, Harold W.
B-L	Strobel, Arthur
EK	Strong, Norman B.
B-L	Taylor, J. E.
EK	Tompkins, Frederick A.
B-L	Trautman, Otto E.
EK	Trivelli, A. P. H.
B-L	Tschume, George G.
EK	Tuttle, Clifton
HE	Tuttle, Fordyce
EK	Vaeth, Wm. C.
U	Warren, Dr. S. L.
B-L	Wason, Robert M.
B-L	Welsch, Maximilian
EK	Westwater, Waldo
U	Wilkins, Dr. T. Russell
EK	Wilsey, Rex B.
ES	Wittel, Otto
B-L	Wittig, Carl
B-L	Wolferts, Oswald H.
ES	Wood, Donald L.
B-L	Yanneck, W. C.
B-L	Zimmerman, George

Der Smekal-Raman Effekt. By K. F. W. KOHLRAUSCH. Julius Springer. June, 1931. Struktur der Materie XII, 85 Figs. p. VIII+392, bound 33.80 RM.

The single fact that from the discovery by Raman of the Smekal-Raman shift in March, 1928 to June, 1931, 413 articles have been published on this phenomenon is alone sufficient justification for Kohlrausch's most successful attempt to organize, review, summarize and correlate the bewildering mass of data so suddenly thrust upon the physicist and chemist. Although the fundamental physical interpretation of the effect as originally outlined by Smekal and as is implied in the Kramers-Heisenberg dispersion formula is indeed quite simple, its detailed application to various dynamical systems has brought forth such unexpected fruits as to make the Raman effect one of the most powerful tools that both the physicist and chemist can use in solving their problems of molecular behavior and structure. To make the results already derived available to the student and to outline paths for future work to the research physicist and chemist has been the aim of this work.

Thus Kohlrausch first introduces the reader to a broad outline of the various features common to all Raman spectra and to the elementary interpretation of the Raman shift, first as given by Smekal and then as it is contained in the Kramer-Heisenberg dispersion formula. The second chapter is very appropriately devoted to a detailed description of the experimental technique that has been found to be suitable for studying the effect as applied to gases, liquids and crystals. This is followed by a discussion of the simplest Raman spectra, namely those involving only a change in the rotational energy of the scattering molecule. Among the spectra described, those of H_2, O_2, N_2 and their importance in answering questions on nuclear spin are presented in detail. General properties of the Raman spectra, as line breadth, coherence of the scattered lines, intensity relations and polarization characteristics of the lines take up the fourth chapter.

Perhaps the most important part of the book is contained in the fifth chapter entitled "Smekal-Raman Effect and Molecular Structure." By means of numerous examples various lines in the Raman spectra of molecules having similar atomic groups are traced to particular chemical bond vibrations in the molecules. Interpretations—at times quite successful—are presented which are based on molecular models vibrating mechanically under the action of either central forces or forces depending on the angles in the geometrical model. These are given for molecules containing two, three, four, and even five atoms, although with successively decreasing certainty of the conclusions. Then in the last chapter some special problems are treated such as the interpretation of the Raman spectra of the benzene molecule, the double bond of the carbon atom, etc. This is followed by an

appendix of 10 sections giving the elements of the classical theory of the interaction of electric charges with radiation, the classical theory of the Raman effect and a more complete discussion of the quantum theory of the effect.

Finally there is presented a classified and complete table of the substances of which the Raman spectra are known; these latter are enumerated and their sources are referred to a complete bibliography which immediately follows the classification table.

Throughout the book numerous diagrams and tables of data are given to illustrate the various points. They can hardly be of a permanent nature because of the inaccuracy of and uncertainty in much of the data and will undoubtedly be replaced by more satisfactory data in future editions. Even then the theory, as to its details, is considerably behind the experiments and this may explain the lack of any extensive theoretical discussions. The book should be heartily welcomed both by those who wish to know the significance of the Raman effect for modern physics and those who are working to add to the vast store of information already correlated and presented in "Der Smekal-Raman Effekt."

MORRIS MUSKAT